Common Agency with Non-Delegation or Imperfect Commitment*

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Abstract

Inspired by Szentes' critique (Szentes (2009)), we study common agency models with non-delegated contracts. In such a setup, we prove that the menu theorem in Peters (2001) holds partially only under some particular information structure, and we use examples to show that it fails generally. Furthermore, we prove a menu-of-menu-with-recommendation theorem in our models. Finally, we show that our results can be easily extended to common agency with imperfect commitment *à la* Bester and Strausz (2000, 2001, 2007).

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1 Introduction

Starting from Bernheim and Whinston (1985, 1986a,b), common agency has been an extremely useful model to analyze strategic interaction between several principals and a common agent in markets. Among many, its applications are *U.S. health care system* (Frandsen and Powell (2019)), *capital tax competition* (Keen and Konrad (2013), Chirinko and Wilson (2018)), *lobbying* (Grossman and Helpman (1994), Dixit, Grossman, and Helpman (1997), Martimort and Semenov (2008), Esteller-More, Galmarini, and Rizzo (2012)), *oligopolistic competition* (d'Aspremont and Ferreira (2010)), and *financial contracting* (Parlour and Rajan (2001), Khalila, Martimort, and Parigic (2007)).

In the classical (one-)principal-(one-or-many-)agent model, the *revelation principle* is a powerful tool for equilibrium analysis.[1] However, it fails in common-agency models.[2] Nevertheless, Peters (2001) and Martimort and Stole (2002) develop a different powerful tool for common-agency models: the *menu theorem*.

A general mechanism offered by principal j to the common agent is a function, $c_j : M_j \rightarrow Y_j$, where M_j is a complicated message space[3], and Y_j is principal j's action space. The interpretation is that the agent could choose any message $m_j \in M_j$, which would pin down j's action $c_j(m_j) \in Y_j$. The menu theorem says that offering c_j is equivalent to offering the menu contract (i.e., a subset of Y_j) described as follows:

$$c_j^{menu} \equiv \{c_j(m_j) \in Y_j : m_j \in M_j\}.$$

By offering c_j^{menu}, principal j lets the agent choose any $y_j \in c_j^{menu}$ and commits to follow his action choice.[4] The intuition of the menu theorem is that, under the general mechanism c_j, if the agent's equilibrium message is $m_j^* \in M_j$, then sending m_j^* under c_j is equivalent to choosing $c_j(m_j^*)$ under c_j^{menu}. Thus, it suffers no loss of generality for principals to offer the menu contracts *only*, which substantially simplifies equilibrium analysis.[5]

[1] The revelation principle says that it suffers no loss of generality for the principal to offer direct mechanisms. In any equilibrium under any general mechanism, each agent's equilibrium strategy depends *only* on his private type. This is equivalent to agents truthfully revealing their private types, and the principal committing to playing agents' equilibrium strategies as dictated by a direct mechanism.

[2] The revelation principle fails when multiple principals coexist, because an agent's equilibrium strategy depends on both his private type and the contracts offered by *all principals*. Thus, it suffers loss of generality to focus on direct mechanisms which depend *only on* private types (but not on other principals' contracts).

[3] For example, $M_j = [0, 1]$. We do no impose restriction on M_j, and it could be much more complicated.

[4] Throughout the paper, we use she to denote a principal and he to denote an agent.

[5] The set of all menu contracts is $2^{Y_j} \diagdown \{\varnothing\}$, while the set of all general mechanisms is $(Y_j)^{M_j}$, and the latter is much more complicated than the former.

The competing mechanism model is a different but related model, in which multiple principals and multiple agents coexist. In this model, though neither the revelation principle nor the menu theorem holds, Yamashita (2010) proves a folk theorem. Both Yamashita's folk theorem and the menu theorem impose a common assumption: each principal offers a delegated contract (i.e., a function, $c_j : M_j \to Y_j$). That is, each principal fully delegates her action to the agent(s) via c_j. Szentes (2009) raises a critique about Yamashita (2010), and argues that principals should offer non-delegated contracts rather than delegated ones. A non-delegated contract is a function, $c_j : M_j \to 2^{Y_j} \diagdown \{\varnothing\}$, i.e., the agents' messages determine only a subset of Y_j, and principal j is free to choose any action in the subset later.[6] In section 2, we review Szentes' critique in details.

Though Szentes' critique is on competing-mechanism games, we find out that the same critique applies to common-agency games. Since the menu theorem is proved under delegated contracts only, this immediately leads to the following two questions: (1) Does the menu theorem still hold if we allow principals to offer non-delegated contracts? (2) If not, how should the menu theorem be adapted? We aim to answer these two questions in this paper.

Another motivation for our study is that common agency with non-delegated contracts is a special case of common agency with imperfect commitment *à la* Bester and Strausz (2000, 2001, 2007). A rigorous relationship between the two models is provided in Section 8. All of our analysis and full characterization in the former model can be easily extended to the latter (see Section 8). The revelation principle in mechanism design with imperfect commitment has been scrutinized by recent papers (e.g., Bester and Strausz (2001), Doval and Skreta (2021)), which provide various economic applications. To the best of our knowledge, this paper is the first one to study the menu theorem (i.e., the counterpart of revelation principle) for common agency with imperfect commitment.

To answer the two questions, we rigorously define a common-agency game without delegation, which turns out to be non-trivial. Such a game consists of three stages.

$$\left(\begin{array}{c} \text{Stage 1: principals simultaneously announce their non-delegated contracts to the agent;} \\ \text{Stage 2: the agent simultaneously sends messages to principals,} \\ \text{which pin down a subset of actions for each principal;} \\ \text{Stage 3: each principal simultaneously chooses an action in the subset.} \end{array}\right)$$

Different from the delegated model in Peters (2001), the announcement and communication structures matter in our model because a principal's action choice at Stage 3 depends on

[6] A non-delegated contract describes scenarios in which a principal is still free to take some actions after a contract is executed. An example is the adjustable rate mortgage (ARM) contract.

what she has observed at the time. Specifically, at Stage 1, each principal may announce her contract to the agent privately or publicly. With public announcement, principals observe all of the contracts, whereas with private announcement, each principal observes only her own contract. Similarly, at Stage 2, the agent may send a message to each principal privately or publicly. With different combinations of announcement and communication protocols, we can define four non-delegated common-agency models.[7]

$$\begin{pmatrix} \text{Model 1: private announcement and private communication;} \\ \text{Model 2: public announcement and private communication;} \\ \text{Model 3: private announcement and public communication;} \\ \text{Model 4: public announcement and public communication} \end{pmatrix}.$$

In Sections 4.1 and 7.1, we use examples to show that the menu theorem fails in Models 2, 3 and 4. For Model 1, Theorem 4 shows that the menu theorem holds partially: every equilibrium allocation under non-delegated contracts is an equilibrium allocation under menu contracts. However, the converse is not true. In this sense, it suffers loss of generality to focus on menu contracts only in Model 1.

Then, how should the menu theorem be adapted under non-delegated contracts? A natural conjecture is that it suffers no loss generality to focus on menu-of-menu contracts, where a menu-of-menu contract is subset $\Lambda_j \subset 2^{Y_j} \diagdown \{\varnothing\}$, and by offering Λ_j, the agent can pick any subset $E_j \in \Lambda_j$ at Stage 2, and principal j commits to playing an action in E_j only at Stage 3. The intuition is that offering a general non-delegated contract $c_j : M_j \longrightarrow 2^{Y_j} \diagdown \{\varnothing\}$ seems to correspond to offering the following menu-of-menu contract

$$c_j^{menu-of-menu} \equiv \left\{ c_j\left(m_j\right) \in 2^{Y_j} \diagdown \{\varnothing\} : m_j \in M_j \right\}.$$

However, this conjecture is incorrect.[8] Rather, we identify two simple contract spaces, which

[7] For the delegated common-agency model in Peters (2001), the announcement and communication protocols do not have impact on equilibria.

[8] The reason is that the messages in a menu-of-menu contract are not rich enough. For instance, in an equilibrium under general non-delegated contracts, suppose types θ and θ' of the agent send two distinct messages m_j and m'_j to principal j at Stage 2, which pin down the same subset $E_j \in 2^{Y_j} \diagdown \{\varnothing\}$, while principal j takes distinct actions $y_j \in E_j$ and $y'_j \in E_j$ at Stage 3, upon observing distinct messages m_j and m'_j, respectively. If we replicate this equilibrium by a menu-of-menu contract, types θ and θ' must choose the same $E_j \in 2^{Y_j} \diagdown \{\varnothing\}$ at Stage 2, and as a result, principal j must take the same action at Stage 3, upon observing the same message E_j from the agent.

are augmented from menu-of-menu contracts in two particular ways:

$$\left(\begin{array}{c} C_j^R\text{: the set of menu-of-menu-with-recommendation contracts for principal } j \text{ (Definition 3)}, \\ C_j^F\text{: the set of menu-of-menu-with-full-recommendation contracts for principal } j \text{ (Definition 4)} \end{array}\right).$$

That is, at Stage 2, the agent not only chooses a subset $c_j^{menu-of-menu}(m_j) \in 2^{Y_j} \diagdown \{\varnothing\}$, but also needs to make a non-binding recommendation $y_j \in c_j^{menu-of-menu}(m_j)$, which would guide each principal j to choose her action at Stage 3. In particular, $C_j^F \subset C_j^R$, and different from the latter, the former requires existence of a subset $c_j^{menu-of-menu}(m_j) \in 2^{Y_j} \diagdown \{\varnothing\}$, such that every $y_j \in c_j^{menu-of-menu}(m_j)$ can be recommended by the agent (i.e., full recommendation on $c_j^{menu-of-menu}(m_j)$).

Our main result (Theorem 2) establishes a menu-of-menu-with-recommendation theorem in Models 1, 2 and 4: it suffers no loss of generality for each principal j to offer a contract in C_j^R *on the equilibrium path*, and to offer contracts in C_j^F *on off-equilibrium paths*. Since both C_j^R and C_j^F are much simpler than the set of all general non-delegated contracts, our theorem substantially simplifies equilibrium analysis in Models 1, 2 and 4.

It is worthy noting that it suffers *loss of generality* for each principal j to offer a contract in C_j^F on the equilibrium path, or to offer contracts in C_j^R on off-equilibrium paths.[9] Our analysis uncovers a feature of our full characterization, which is not shared by the menu theorem: asymmetric contract spaces between on and off the equilibrium path. Specifically, to prove our full characterization, we need to replicate an equilibrium on the general contract space with an equilibrium on a simple contract space: on the equilibrium path, we just mimic *one contract profile* in the former space with one contract profile in the latter space, and we show C_j^R suffices for the latter space (but C_j^F does not); on off-equilibrium paths, we mimic *all possible contract profiles* (due to all possible deviations) in the former space with those in the latter space, and we show C_j^F suffices for the latter space (but C_j^R does not). We embed this subtle strategic difference into the different requirements in the definitions of C_j^R and C_j^F discussed above.

In Section 7.1, we use an example to show that the menu-of-menu-with-recommendation theorem fails in Model 3. Thus, model 3 needs a more complicated full characterization, which is also provided in Section 7.2.

The remainder of the paper proceeds as follows: we review Szentes' critique in Section 2, and describe the model in Section 3; we study the menu theorem in Section 4, and propose

[9]The intuition is similar to that in Footnote 8: contracts in C_j^F and C_j^R do not have enough messages to describe general contracts on and off the equilibrium path, respectively.

two simpler contract spaces in Section 5; we present our main results in Sections 6 and 7; we consider imperfect commitment in Section 8 and conclude in Section 9.

2 Szentes' critique

Consider the following example in Szentes (2009). There are two principals and three agents with one payoff-relevant state (i.e., complete information). Each principal chooses one of the two actions, H and T, and principals' payoffs are listed as follows.

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

All agents are indifferent among all action profiles.

Following Yamashita (2010), Szentes (2009) focuses on pure strategies. The min max value of each principal's payoffs is 1, whereas the max min value is -1. For instance, (T, H) induces the max min value for principal 1 (i.e., the row player). By Yamashita (2010), (T, H) can be induced by an equilibrium. Specifically, on the equilibrium path, each principal j offers the delegated contract $c_j : \{H, T\} \times \{H\} \times \{H\} \to \{H, T\}$ with

$$c_j (H, H, H) = H \text{ and } c_j (T, H, H) = T.$$

That is, principals invite agent 1 only to vote regarding "H vs T,"[10] and principals follow agent 1's recommendation. Upon receiving such a contract profile on the equilibrium path, agent 1 votes T and H for principals 1 and 2, respectively, which induces (T, H).

To show this is an equilibrium, consider principal 1's deviation to a delegated contract,[11] and let $\widehat{m}$ denote a message profile that induces an action $y_1 \in \{H, T\}$. Given this unilateral deviation, it is a continuation equilibrium that the agents send $\widehat{m}$ to principal 1 and agent 1 recommends $y_2 \in \{H, T\} \diagdown \{y_1\}$ to principal 2, because agents are indifferent among all action profiles. This still induces the max min value for principal 1, i.e., not a profitable deviation for principal 1.

However, Szentes (2009) doubts the legitimacy of (T, H) being an equilibrium outcome. Szentes (2009) argues that, in any reasonable equilibrium, every principal must achieve at least her min max value, because she can always opt out of this contract game

[10] The message sets for agents 2 and 3 are degenerate (i.e., $|\{H\}| = 1$), and their votes are not informative.

[11] Principal 2 achieves the maximal payoff under (T, H), i.e., her incentive compatibility holds.

and achieves her min max value in the ensuing equilibrium.[12] Furthermore, Szentes (2009) argues that principals should offer non-delegated contracts, and proves that each principal indeed achieves at least her min max value under non-delegated contracts.

Though there are multiple agents in the example above, the same logic still applies if we delete agents 2 and 3 from the example. Therefore, Szentes' critique also applies to the common-agency model.

The common assumption in Yamashita (2010) and Peters (2001) is: principals are *not allowed* to offer non-delegated contracts, whereas, Szentes' critique is: if principals' equilibrium payoffs increase by offering non-delegated contracts, there is no reason to forbid them.

One argument against Szentes' critique is that Szentes' example does not work if we allow for mixed-strategies, which would make the min max value be equal to the max min value. Given mixed-strategies, though Szentes' example fails, Szentes' point remains valid for *common-agency models*, which is supported by the example in Section 4.2.1. In this example, there is an equilibrium under non-delegated contracts, which strictly Pareto dominates any mixed-strategy equilibrium under delegated contracts.

3 Preliminaries

3.1 Primitives

A single agent privately observes her type $\theta \in \Theta$, which is drawn from a common prior $p \in \Delta(\Theta)$ with full support. Let $\mathcal{J} \equiv \{1, \ldots, J\}$ be the set of principals, with $J \geq 2$. Each principal j takes an action $y_j \in Y_j$. Let $Y \equiv \times_{j \in \mathcal{J}} Y_j$. Principal j's utility function is $v_j : Y \times \Theta \to \mathbb{R}$. The agent's utility function is $u : Y \times \Theta \to \mathbb{R}$. We assume $|Y \times \Theta| < \infty$.

For each $j \in \mathcal{J}$, principal j's contract is a function $c_j : M_j^{\mathcal{A}} \to \mathcal{A}_j$, where $M_j^{\mathcal{A}}$ is an infinite set of messages that the agent can send to principal j and

$$\mathcal{A}_j^{delegated} \equiv \{\{y_j\} : y_j \in Y_j\}, \mathcal{A}_j^{non-delegated} \equiv 2^{Y_j} \diagdown \{\varnothing\},$$
$$\mathcal{A}_j \in \left\{\mathcal{A}_j^{delegated}, \mathcal{A}_j^{non-delegated}\right\}.$$

$M_j^{\mathcal{A}}$ can be very general and we do not impose any other restriction on it. Let $M^{\mathcal{A}} \equiv$

[12] See more discussions in Peters (2014).

$\times_{k\in\mathcal{J}}M_k^{\mathcal{A}}$ and $M_{-j}^{\mathcal{A}} \equiv \times_{k\in\mathcal{J}\diagdown\{j\}}M_k^{\mathcal{A}}$.[13] If $\mathcal{A}_j = \mathcal{A}_j^{delegated}$, principal j delegates her action to the agent, i.e., the agent's message fully determines j's action. If $\mathcal{A}_j = \mathcal{A}_j^{non-delegated}$, principal j has to strategically choose an action in $c_j(m_j) \in 2^{Y_j}\diagdown\{\varnothing\}$ after receiving a message m_j. Define $\mathcal{A} \equiv \times_{k\in\mathcal{J}}\mathcal{A}_k$ and $\mathcal{A}_{-j} \equiv \times_{k\in\mathcal{J}\diagdown\{j\}}\mathcal{A}_k$. Let $C_j^{\mathcal{A}} \equiv (\mathcal{A}_j)^{M_j^{\mathcal{A}}}$ be the set of all possible contracts available to principal j, $C^{\mathcal{A}} \equiv \times_{k\in\mathcal{J}}C_k^{\mathcal{A}}$ and $C_{-j}^{\mathcal{A}} \equiv \times_{k\in\mathcal{J}\diagdown\{j\}}C_k^{\mathcal{A}}$.

3.2 A generic game

We will consider different contract spaces (and message spaces). Principals and the agent will play a generic game under different contract spaces, which is defined as follow.

For each $j \in \mathcal{J}$, let M_j be a generic set of messages that the agent may send to principal j, and a contract of principal j is $c_j : M_j \to \mathcal{A}_j$. Thus, $C_j \equiv (\mathcal{A}_j)^{M_j}$ is a generic contract space for principal j, which may represent a different contract space (e.g., C_j^P, C_j^R, C_j^F defined later), besides $C_j^{\mathcal{A}}$ in Section 3.1. Denote $C \equiv \times_{k\in\mathcal{J}}C_k$ and $M \equiv \times_{k\in\mathcal{J}}M_k$.

Throughout this subsection, we fix a generic profile (M, C) to define a generic game.

3.2.1 Models and timeline

When principal j chooses her action in the set $c_j(m_j)$, her decision depends not only on the message she receives (i.e., m_j), but also on what she observes regarding the contracts offered by the other principals and the messages that the agent sends to all principals.

Let $\Gamma \equiv \left[\Gamma_k : C \to 2^C\right]_{k\in\mathcal{J}}$ denote a potential announcement structure. We focus on two structures: (1) public announcement (denoted by $\Gamma^{public} = \left(\Gamma_k^{public}\right)_{k\in\mathcal{J}}$) with

$$\Gamma_k^{public}(c_k, c_{-k}) = \{(c_k, c_{-k})\}, \forall k \in \mathcal{J}, \forall (c_k, c_{-k}) \in C, \tag{1}$$

and (2) private announcement (denoted by $\Gamma^{private} = \left(\Gamma_k^{private}\right)_{k\in\mathcal{J}}$) with

$$\Gamma_k^{private}(c_k, c_{-k}) = \{c_k\} \times C_{-k}, \forall k \in \mathcal{J}, \forall (c_k, c_{-k}) \in C. \tag{2}$$

I.e., principals observe all of the contracts offered under public announcement, whereas, under private announcement, each principal knows only her own contract.

[13] We assume $M^{\mathcal{A}} \equiv M^{\mathcal{A}^{delegated}} \equiv M^{\mathcal{A}^{non-delegated}}$, and we use the superscript $\mathcal{A}$ simply to distinguish it from M^R and M^F, which will be defined later.

Similarly, let $\Psi \equiv \left[\Psi_k : M \to 2^M\right]_{k\in\mathcal{J}}$ denote a communication structure. We focus on two structures: (1) public communication (denoted by $\Psi^{public} = \left(\Psi_k^{public}\right)_{k\in\mathcal{J}}$) with

$$\Psi_k^{public}(m_k, m_{-k}) = \{(m_k, m_{-k})\}, \forall k \in \mathcal{J}, \forall (m_k, m_{-k}) \in M, \tag{3}$$

and (2) private communication (denoted by $\Psi^{private} = \left(\Psi_k^{private}\right)_{k\in\mathcal{J}}$) with

$$\Psi_k^{private}(m_k, m_{-k}) = \{m_k\} \times M_{-k}, \forall k \in \mathcal{J}, \forall (m_k, m_{-k}) \in M. \tag{4}$$

I.e., under public communication, all principals observe all messages, whereas, under private communication, each principal observes only the message she receives.

Thus, a model is characterized by a tuple $\langle \mathcal{A}, \Gamma, \Psi \rangle$, where

$$\mathcal{A} \in \left\{\mathcal{A}^{delegated}, \mathcal{A}^{non-delegated}\right\}, \ \Gamma \in \left\{\Gamma^{private}, \ \Gamma^{public}\right\}, \ \Psi \in \left\{\Psi^{private}, \ \Psi^{public}\right\}.$$

Given a model $\langle \mathcal{A}, \ \Gamma, \ \Psi \rangle$, the game proceeds according to the following timeline.

1. Before the game starts, Nature draws the agent's type according to the common prior $p \in \Delta(\Theta)$ and the realized type is the agent's private information;

2. At Stage 1, each principal $j \in \mathcal{J}$ simultaneously offers a contract $c_j \in C_j$ to the agent. The agent observes $c \equiv (c_1, \ldots, c_J)$, whereas each principal $j \in \mathcal{J}$ knows that only a contract profile in $\Gamma_j(c)$ is possibly chosen by the principals.

3. At Stage 2, the agent sends messages $m \equiv (m_1, \ldots, m_J) \in M$, one for each principal. Each principal $j \in \mathcal{J}$ knows that only a message profile in $\Psi_j(m)$ is possibly sent by the agent to the principals;

4. At Stage 3, each principal $j \in \mathcal{J}$ simultaneously chooses an action in $c_j(m_j)$;

5. Finally, payoffs are realized.

In the model of Peters (2001), the agent also chooses an effort. For simplicity, we choose not to include the agent's effort in our model. Another reason for this modeling choice is that the menu theorem already fails in such a simple model. In Han and Xiong (2022), we show how our results can be extended to a model with the agent's effort.

3.2.2 Strategies

At Stage 1, each principal j chooses $c_j \in C_j$. At Stage 2, the agent chooses a function $s \equiv [s_k : C \times \Theta \to M_k]_{k \in \mathcal{J}}$. Let S_k be the set of all possible $s_k : C \times \Theta \to M_k$. Denote $s \equiv (s_1, \ldots, s_J) \in S \equiv \times_{k \in \mathcal{J}} S_k$ and $s_{-j} \in S_{-j} \equiv \times_{k \in \mathcal{J} \diagdown \{j\}} S_k$. Denote $s(c, \theta) = (s_1(c, \theta), \ldots, s_J(c, \theta))$ for each $(c, \theta) \in C \times \Theta$. At Stage 3, each principal j's chooses a function $t_j : \Gamma_j(C) \times \Psi_j(M) \to Y_j$ such that

$$t_j [\Gamma_j (c_j, c_{-j}), \Psi_j (m_j, m_{-j})] \in c_j(m_j), \forall [(c_j, c_{-j}), (m_j, m_{-j})] \in C \times M,$$

and let T_j be the set of all such functions. Denote $T \equiv (T_k)_{k \in \mathcal{J}}$ and $T_{-j} \equiv (T_k)_{k \in \mathcal{J} \diagdown \{j\}}$.

Given any $(c, s, t) \in C \times S \times T$, the utility for the agent of type θ is

$$U(c, s, t, \theta) \equiv u(t_1(\Gamma_1(c), \Psi_1(s(c, \theta))), \ldots, t_J(\Gamma_J(c), \Psi_J(s(c, \theta))), \theta),$$

and principal j's expected utility is

$$V_j(c, s, t) \equiv \sum_{\theta \in \Theta} p(\theta) \times v_j(t_1(\Gamma_1(c), \Psi_1(s(c, \theta))), \ldots, t_J(\Gamma_J(c), \Psi_J(s(c, \theta))), \theta).$$

3.2.3 Legitimate beliefs

At Stage 3, principal j must form a belief on $(C \times M \times \Theta)$ conditional on $\Gamma_j(c)$ she observes at Stage 1 and $\Psi_j(m)$ she observes at Stage 2. It is described by a function $b_j : \Gamma_j(C) \times \Psi_j(M) \to \Delta(C \times M \times \Theta)$. Given belief b_j, principal j's expected utility conditional on $(\alpha_j, \beta_j) \in \Gamma_j(C) \times \Psi_j(M)$ is

$$V_j(t_j, t_{-j} | \alpha_j, \beta_j, b_j) \equiv \int_{b_j(\alpha_j, \beta_j)} v_j(t_1(\Gamma_1(c), \Psi_1(m)), \ldots, t_J(\Gamma_J(c), \Psi_J(m)), \theta) \, d(c, m, \theta).$$

For each principal j's belief, we apply Bayes' rule if and only if she cannot confirm that the other players have deviated from an equilibrium. Given (c, s) being played in an equilibrium, define $\mathcal{B}_j^{(c,\,s)}$ as the set of principal j' valid beliefs. For every $j \in \mathcal{J}$,

$$\mathcal{B}_j^{(c,\,s)} \equiv \left\{ \begin{array}{c} b_j : \Gamma_j(C) \times \Psi_j(M) \to \Delta(C \times M \times \Theta) \text{ such that} \\ b_j[\Gamma_j(c'), \Psi_j(m)](\Gamma_j(c') \times \Psi_j(m) \times \Theta) = 1, \forall (c', m) \in C \times M, \\ \\ \left(\begin{array}{c} \forall c_j' \in C_j, \ \forall \theta \in \Theta, \\ \text{set } \beta_j = \Psi_j\left(s\left(c_j', c_{-j}, \theta\right)\right), \text{ and we have} \\ b_j\left[\Gamma_j\left(c_j', c_{-j}\right), \ \beta_j\right]\left(c_j', c_{-j}, \ s\left(c_j', c_{-j}, \theta\right), \ \theta\right) = \dfrac{p(\theta)}{\sum\limits_{\theta'' \in \left\{\theta' \in \Theta: \ \beta_j = \Psi_j\left(s\left(c_j', c_{-j}, \theta'\right)\right)\right\}} p(\theta'')} \end{array} \right) \end{array} \right\}. \tag{5}$$

First,

$$b_j[\Gamma_j(c'), \Psi_j(m)](\Gamma_j(c') \times \Psi_j(m) \times \Theta) = 1, \forall (c', m) \in C \times M \tag{6}$$

in (5) is the classic perfect recall condition. For instance, given perfect announcement, if principal j observes $c' \in C$ at Stage 1, j must believe in c' with probability 1 at Stage 3 (even on off-equilibrium paths).[14]

Second, we will adopt the solution concept of (weak) Perfect Bayesian equilibrium, and hence, players will use Bayes' rule to update their beliefs whenever possible. As usual, when one principal deviates unilaterally, she assumes that the other players follows the equilibrium strategy profile. This is rigorously described in the set $\mathcal{B}_j^{(c,\,s)}$ in (5). Specifically, $\mathcal{B}_j^{(c,\,s)}$ contains any belief function b_j which satisfies the following condition. Given (c, s) being played in an equilibrium, suppose principal j unilaterally deviates to $c_j' \in C_j$. If j observes $\Gamma_j\left(c_j', c_{-j}\right)$ (i.e., j cannot confirm that principals $-j$ have deviated from (c, s)) and observes $\beta_j = \Psi_j\left(s\left(c_j', c_{-j}, \theta\right)\right)$ for some $\theta \in \Theta$ (i.e., j cannot confirm that agents have deviated from (c, s)), principal j believes that principals $-j$ have offered c_{-j}, and the agent has followed s. As a result, the following set contains all possible states,

$$\left\{\theta' \in \Theta : \ \beta_j = \Psi_j\left(s\left(c_j', c_{-j}, \theta'\right)\right)\right\},$$

and by Bayes' rule, j's updated belief is

$$b_j\left[\Gamma_j\left(c_j', c_{-j}\right), \ \beta_j\right]\left(c_j', c_{-j}, \ s\left(c_j', c_{-j}, \theta\right), \ \theta\right) = \frac{p(\theta)}{\sum\limits_{\theta'' \in \left\{\theta' \in \Theta: \ \beta_j = \Psi_j\left(s\left(c_j', c_{-j}, \theta'\right)\right)\right\}} p(\theta'')}.$$

If principal j can confirm that either principals $-j$ or the agent have deviated from (c, s), we impose no requirement on $b_j \in \mathcal{B}_j^{(c,\,s)}$, because this happens with probability 0 in an equilibrium, and Bayes rule does not apply.

[14] If we do not impose the perfect recall condition, our results and proofs remain true.

3.2.4 Perfect Bayesian equilibrium

For simplicity, we adopt the solution concept of pure-strategy perfect Bayesian equilibrium. For simplicity, we just call it an equilibrium. Our results can be extended to mixed-strategy equilibria (See Han and Xiong (2022)).

Definition 1 *Given (C, M) in a model $\langle \mathcal{A},\ \Gamma,\ \Psi \rangle$, $(c, s, t) \in C \times S \times T$ is a C-equilibrium if*

$$\exists \left(b_k\right)_{k \in \mathcal{J}} \in \times_{k \in \mathcal{J}} \mathcal{B}_k^{(c,\ s)},$$

such that (i) for every $j \in \mathcal{J}$,

$$V_j\left(\left(c_j, c_{-j}\right), s, t\right) \geq V_j\left(\left(c_j', c_{-j}\right), s, t\right),\ \forall c_j' \in C_j,$$

and (ii) for every $\theta \in \Theta$,

$$U\left(c', s, t, \theta\right) \geq U\left(c', s', t, \theta\right),\ \forall j \in \mathcal{J},\ \forall \left(c', s'\right) \in C \times S,$$

and (iii) for every $j \in \mathcal{J}$,

$$V_j\left(t_j, t_{-j} | \alpha_j, \beta_j, b_j\right) \geq V_j\left(t_j', t_{-j} | \alpha_j, \beta_j, b_j\right),\ \forall t_j' \in T_j,\ \forall \left(\alpha_j, \beta_j\right) \in \Gamma_j\left(C\right) \times \Psi_j\left(M\right).$$

Let $\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle\text{-}C}$ denote the set of C-equilibria in the model $\langle \mathcal{A},\ \Gamma,\ \Psi \rangle$.

3.2.5 Allocation

Let $Z \equiv \times_{k \in \mathcal{J}} \left[Z_k : \Theta \rightarrow Y_k\right]$ denote the set of allocations. Given any $(c, s, t) \in C \times S \times T$, define $z^{(c,s,t)} : \Theta \rightarrow Y$ as

$$z^{(c,s,t)}\left(\theta\right) = \left[z_k^{(c,s,t)}\left(\theta\right)\right]_{k \in \mathcal{J}} = \left[t_k\left(c_k, s_k\left(c, \theta\right)\right)\right]_{k \in \mathcal{J}},\ \forall \theta \in \Theta,$$

i.e., $z^{(c,s,t)}$ is the allocation induced by (c, s, t). Define

$$Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle\text{-}C}} \equiv \left\{z^{(c,s,t)} \in Z : (c, s, t) \in \mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle\text{-}C}\right\}, \tag{7}$$

i.e., $Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle\text{-}C}}$ is the set of all C-equilibrium allocations in the model $\langle \mathcal{A},\ \Gamma,\ \Psi \rangle$.

3.3 The primitive contract space and the goal

The primitive contract space is $C^{\mathcal{A}}$ in Section 3.1, and our goal is a simple full characterization of $Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle\text{-}C^{\mathcal{A}}}}$. Given $\mathcal{A} = \mathcal{A}^{delegated}$, the menu theorem in Peters (2001) has achieved this goal, which will be reviewed in Section 4.1. Thus, given $\mathcal{A} = \mathcal{A}^{non-delegated}$, we aim to provide a simple full characterization of $Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle\text{-}C^{\mathcal{A}}}}$ for each model $\langle \mathcal{A},\ \Gamma,\ \Psi \rangle$.

4 The menu theorem in Peters (2001)

4.1 The menu theorem for delegated contracts

The menu theorem in Peters (2001) is established under delegated contracts, i.e., $\mathcal{A} = \mathcal{A}^{delegated}$. A menu contract introduced in Peters (2001) is a function, $c_j : E_j \to Y_j$, with

$$E_j \in 2^{Y_j} \diagdown \{\varnothing\} \text{ and } c_j(y_j) = y_j, \forall y_j \in E_j.$$

Let C_j^P denote the set of all menu contracts for principal j, $C^P \equiv \times_{k \in \mathcal{J}} C_k^P$, and $C_{-j}^P \equiv \times_{k \in \mathcal{J} \diagdown \{j\}} C_k^P$. Let $M_j^P \equiv Y_j$ denote the set of messages used in all possible menus, $M^P \equiv \times_{k \in \mathcal{J}} M_k^P$, and $M_{-j}^P \equiv \times_{k \in \mathcal{J} \diagdown \{j\}} M_k^P$.

Theorem 1 (The menu Theorem, Peters (2001)) *We have*

$$Z^{\mathcal{E}^{\left\langle \mathcal{A}^{delegated}, \Gamma, \Psi \right\rangle \text{-} C^{\mathcal{A}^{delegated}}}} = Z^{\mathcal{E}^{\left\langle \mathcal{A}^{delegated}, \Gamma, \Psi \right\rangle \text{-} C^{P}}}, \forall \langle \Gamma, \Psi \rangle \in \left\{\Gamma^{private}, \Gamma^{public}\right\} \times \left\{\Psi^{private}, \Psi^{public}\right\}. \tag{8}$$

With delegated contracts (i.e., $\mathcal{A} = \mathcal{A}^{delegated}$), the announcement and communication structures do not have impact on equilibiria[15].

The implication of Theorem 1 is that, given delegated contracts, it suffers no loss of generality for principals to offer menus both on and off the equilibrium path. Since C^P is a much simpler set than $C^{\mathcal{A}}$, this result substantially simplifies the characterization of equilibrium allocations.

4.2 Failure of the menu theorem for non-delegated contracts

(8) in Theorem 1 can be dissected into two parts:

$$Z^{\mathcal{E}^{\left\langle \mathcal{A}^{delegated}, \Gamma, \Psi \right\rangle \text{-} C^{\mathcal{A}^{delegated}}}} \supset Z^{\mathcal{E}^{\left\langle \mathcal{A}^{delegated}, \Gamma, \Psi \right\rangle \text{-} C^{P}}} \text{ and } Z^{\mathcal{E}^{\left\langle \mathcal{A}^{delegated}, \Gamma, \Psi \right\rangle \text{-} C^{\mathcal{A}^{delegated}}}} \subset Z^{\mathcal{E}^{\left\langle \mathcal{A}^{delegated}, \Gamma, \Psi \right\rangle \text{-} C^{P}}}.$$

Does the menu theorem extends to non-delegated contracts, i.e., do the following hold?

[15] Different announcement and communication structures lead to different information for principals *only after they offer their contracts.* With delegated contracts, principals do not make any strategic decision after offerring their contracts, and hence, the information (induced by different the announcement and communication structures) is irrelevant.

$$Z^{\mathcal{E}^{\left\langle \mathcal{A}^{non-delegated},\ \Gamma,\ \Psi\right\rangle\text{-}C^{\mathcal{A}^{non-delegated}}}} \supset Z^{\mathcal{E}^{\left\langle \mathcal{A}^{non-delegated},\ \Gamma,\ \Psi\right\rangle\text{-}C^{P}}}, \tag{9}$$

$$Z^{\mathcal{E}^{\left\langle \mathcal{A}^{non-delegated},\ \Gamma,\ \Psi\right\rangle\text{-}C^{\mathcal{A}^{non-delegated}}}} \subset Z^{\mathcal{E}^{\left\langle \mathcal{A}^{non-delegated},\ \Gamma,\ \Psi\right\rangle\text{-}C^{P}}}. \tag{10}$$

Let us modify the example in Section 2 by deleting agents 2 and 3. In this examples, principals offers menu contracts to agent 1, and for the equilibrium described in Section 2 (i.e. (T, H)), principal 1 achieves the max min value. However, it is straightforward to see that, in any equilibrium with non-delegated contracts, principal 1 must achieve at least her min max value. This shows failure of (9) regardless of $\langle\Gamma,\ \Psi\rangle$.

If (10) holds, we still have a weak sense of the menu theorem: $Z^{\mathcal{E}^{\left\langle \mathcal{A}^{non-delegated},\ \Gamma,\ \Psi\right\rangle\text{-}C^{P}}}$ serves as a superset of $Z^{\mathcal{E}^{\left\langle \mathcal{A}^{non-delegated},\ \Gamma,\ \Psi\right\rangle\text{-}C^{\mathcal{A}^{non-delegated}}}}$. With $\langle\Gamma^{private},\ \Psi^{private}\rangle$, we will prove (10) in Section 6 (i.e., Theorem 4).

In this section, we focus on $\left\langle\Gamma^{public},\ \Psi^{public}\right\rangle$ and $\left\langle\Gamma^{public},\ \Psi^{private}\right\rangle$, and we use examples to show failure of (10) in Sections 4.2.1 and 4.2.2, respectively. The example in Section 7.1 implies failure of (10) under $\left\langle\Gamma^{private},\ \Psi^{public}\right\rangle$.[16]

4.2.1 Public announcement and public communication

Given public announcement and public communication, consider the following example.

$$\Theta = \left\{\theta^1, \theta^4\right\}, \mathcal{J} = \{j_1, j_2\}, Y_{j_1} = Y_{j_2} = \{1, 2, 3, 4\}.$$

The common prior is $p\left(\theta^1\right) = p\left(\theta^4\right) = 1/2$. The preference is defined as follows.

$$v_{j_1}\left[(y_{j_1}, y_{j_2}), \theta^1\right] = v_{j_2}\left[(y_{j_1}, y_{j_2}), \theta^1\right] = \begin{cases} 8, & \text{if } (y_{j_1}, y_{j_2}) = (1,1), \\ 1, & \text{if } (y_{j_1}, y_{j_2}) \neq (1,1) \text{ and } (y_{j_1} + y_{j_2}) \text{ is even}, \\ -1, & \text{if } (y_{j_1} + y_{j_2}) \text{ is odd}, \end{cases}$$

$$v_{j_1}\left[(y_{j_1}, y_{j_2}), \theta^4\right] = v_{j_2}\left[(y_{j_1}, y_{j_2}), \theta^4\right] = \begin{cases} 8, & \text{if } (y_{j_1}, y_{j_2}) = (4,4), \\ 1, & \text{if } (y_{j_1}, y_{j_2}) \neq (4,4) \text{ and } (y_{j_1} + y_{j_2}) \text{ is even}, \\ -1, & \text{if } (y_{j_1} + y_{j_2}) \text{ is odd}, \end{cases}$$

$$u\left[(y_{j_1}, y_{j_2}), \theta^1\right] = u\left[(y_{j_1}, y_{j_2}), \theta^4\right] = \begin{cases} 1, & \text{if } (y_{j_1}, y_{j_2}) = (1,4), \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

[16] The example in Section 7.1 shows a stronger point.

For each $j \in \mathcal{J}$, fix any two distinct messages, m_j^1 and m_j^4, and define a contract, $c_j^* : M_j^{\mathcal{A}} \to 2^{Y_j}$ as follows.

$$c_j^*(m_j) = \begin{cases} \{1,2\}, & \text{if } m_j = m_j^1, \\ \{3,4\}, & \text{if } m_j = m_j^4, \\ \{3,4\}, & \text{otherwise} \end{cases} .$$

Consider the following equilibrium.

$$\text{on the equilibrium path: } \begin{pmatrix} \text{principal } j_1 \text{ offers } c_{j_1}^*, \\ \text{principal } j_2 \text{ offers } c_{j_2}^*, \end{pmatrix}$$

$$\text{on the equilibrium path: } \begin{pmatrix} \text{at state } \theta^1: \text{ the agent sends } m_{j_1}^1 \text{ to } j_1 \text{ and } m_{j_2}^1 \text{ to } j_2, \\ \text{at state } \theta^4: \text{ the agent sends } m_{j_1}^4 \text{ to } j_1 \text{ and } m_{j_2}^4 \text{ to } j_2, \end{pmatrix}$$

$$\text{on the equilibrium path: } \begin{pmatrix} \text{the principals chooses } (1,1), \text{ upon receiving } \left(m_{j_1}^1, m_{j_2}^1\right), \\ \text{the principals chooses } (4,4), \text{ upon receiving } \left(m_{j_1}^4, m_{j_2}^4\right). \end{pmatrix}$$

On the equilibrium path, the induced outcome is $(1,1)$ at state θ^1 and $(4,4)$ at state θ^4, i.e., principals achieve their maximal utility at both states, and hence their incentive compatibility at Stage 1 holds. To sustain this as an equilibrium, the agent and principals take the following (behavior) strategies at Stages 2 and 3.

At Stage 2, the agent takes $s \equiv [s_k : C^{\mathcal{A}} \times \Theta \to M_k^{\mathcal{A}}]_{k \in \mathcal{J}}$ such that

$$s(c,\theta) = \begin{cases} m, & \text{if } \exists m \in M^{\mathcal{A}} \text{ such that } c(m) = \{1\} \times \{4\}, \\ \left(m_{j_1}^1, m_{j_2}^4\right), & \text{otherwise.} \end{cases}, \forall (c,\theta) \in C^{\mathcal{A}} \times \Theta; \tag{12}$$

At Stage 3, j_1 takes $t_{j_1} : C^{\mathcal{A}} \times M^{\mathcal{A}} \to Y_{j_1}$ and j_2 takes $t_{j_2} : C^{\mathcal{A}} \times M^{\mathcal{A}} \to Y_{j_2}$ such that

$$c(m) = \{1\} \times \{4\} \Rightarrow t_{j_1}(c,m) = 1 \text{ and } t_{j_2}(c,m) = 4, \tag{13}$$

and

$$c(m) \neq \{1\} \times \{4\} \Rightarrow [t_{j_1}(c,m), t_{j_2}(c,m)] \in \arg \max_{\left(y_{j_1}, y_{j_2}\right) \in c(m) \setminus \{(1,4)\}} v_{j_1}\left[\left(y_{j_1}, y_{j_2}\right), \theta^1\right]. \tag{14}$$

Since principals have the same utility, they can coordinate their actions under public announcement and public communication. At Stage 3, if the agent's message pins down principals' action profile as $(1,4)$, (13) says that principals would take $(1,4)$, and otherwise,

(14) says that principals would believe the state is θ^1 and take an optimal action profile in $c(m) \diagdown \{(1,4)\}$. As a result, incentive compatibility of principals at Stage 3 holds.

At Stage 2, (12) says that the agent would send a message to pin down $\{1\} \times \{4\}$ for principals if such a message exists, and thus achieve his maximal utility. If there is no message that can pin down $\{1\} \times \{4\}$ for principals, the agent is indifferent among all of the messages, because in this case, by (14), principals would choose an action profile different from $(1,4)$. By (11), the agent is indifferent between any action profiles in $Y \diagdown \{(1,4)\}$. Therefore, incentive compatibility of the agent at Stage 2 holds.

However, the equilibrium allocation described above cannot be replicated by menu contracts. We prove this by contradiction. Suppose that we can do it. Then, we must get $(1,1)$ at state θ^1 and get $(4,4)$ at state θ^4. Thus, both principals' equilibrium menu contracts must include both 1 and 4. Then, $(1,1)$ cannot be achieved at state θ^1, because the agent would deviate to choose 1 from principal j_1's menu and 4 from principal j_2's menu, which achieves the maximal utility for the agent at state θ^1.

4.2.2 public announcement and private communication

In this subsection, we focus on public announcement and private communication with

$$\Theta = \{\theta\}, \mathcal{J} = \{j_1, j_2\}, Y_{j_1} = Y_{j_2} = \{0,1\},$$

i.e., it is a complete-information setup. Principals' preference are listed as follows.

$(v_{j_1}[(y_{j_1}, y_{j_2}), \theta], \ v_{j_2}[(y_{j_1}, y_{j_2}), \theta]):$	$y_{j_2} = 0$	$y_{j_2} = 1$
$y_{j_1} = 0$	$0, 0$	$8, -8$
$y_{j_1} = 1$	$0, 0$	$1, 1$

Furthermore,

$$v_{j_1}[(y_{j_1}, y_{j_2}), \theta] = u[(y_{j_1}, y_{j_2}), \theta], \forall [(y_{j_1}, y_{j_2}), \theta] \in Y_{j_1} \times Y_{j_2} \times \Theta.$$

Fix any $\left(m^{**}_{j_1}, m^{**}_{j_2}\right) \in M^{\mathcal{A}}_{j_1} \times M^{\mathcal{A}}_{j_2}$, and consider the following equilibrium.

$$\text{on the equilibrium path: } \begin{pmatrix} \text{principal } j_1 \text{ offers } c^{**}_{j_1} : M^{\mathcal{A}}_{j_1} \to 2^{Y_{j_1}} \text{ with } c^{**}_{j_1}(m_{j_1}) = \{1\}, \forall m_{j_1} \in M^{\mathcal{A}}_{j_1} \\ \text{principal } j_2 \text{ offers } c^{**}_{j_2} : M^{\mathcal{A}}_{j_2} \to 2^{Y_{j_2}} \text{ with } c^{**}_{j_2}(m_{j_2}) = \{0,1\}, \forall m_{j_2} \in M^{\mathcal{A}}_{j_2} \end{pmatrix},$$

$$\text{on the equilibrium path: } \begin{pmatrix} \text{the agent sends } m^{**}_{j_1} \text{ to principal } j_1 \\ \text{the agent sends } m^{**}_{j_2} \text{ principal } j_2 \end{pmatrix},$$

$$\text{on the equilibrium path: } \begin{pmatrix} \text{principal } j_1 \text{ chooses 1 from the subset } \{1\} \\ \text{principal } j_2 \text{ chooses 1 from the subset } \{0,1\} \end{pmatrix}.$$

To sustain this as an equilibrium, the agent follows the table below to send messages at Stage 2, and principals also follow this table to play continuation equilibrium at Stage 3.

Ranking:	1	2	3	4	5	6	7	8	9
$c_{j_1}(m_{j_1}) \times c_{j_2}(m_{j_2}) =$	$\{0\}\times\{1\}$	$\{0,1\}\times\{1\}$	$\{1\}\times\{1\}$	$\{1\}\times\{0,1\}$	$\{0,1\}\times\{0,1\}$	$\{0\}\times\{0,1\}$	$\{0\}\times\{0\}$	$\{1\}\times\{0\}$	$\{0,1\}\times\{0\}$
continuation equilibrium at Stage 3	$(0,1)$	$(0,1)$	$(1,1)$	$(1,1)$	$(0,0)$	$(0,0)$	$(0,0)$	$(1,0)$	$(1,0)$
$v_{j_1} = u =$	8	8	1	1	0	0	0	0	0

That is, there are 9 non-empty subsets of $Y_{j_1} \times Y_{j_2}$, and for each subset (listed at Row 2) that is pinned down by the agent's messages at Stage 3, we let principals play the corresponding continuation (Nash) equilibrium (listed at Row 3). Row 4 lists the payoffs of these continuation equilibria for the agent and principal j_1. Furthermore, row 1 ranks these continuation equilibria (and the corresponding subsets of $Y_{j_1} \times Y_{j_2}$), according to the agent's payoffs (at Row 4). Clearly, principals' incentive compatibility hold at Stage 3.

We say a contract profile (c_{j_1}, c_{j_2}) is ranked n-th if and only if n is the smallest integer such that there exists (m_{j_1}, m_{j_2}) and $c_{j_1}(m_{j_1}) \times c_{j_2}(m_{j_2})$ is ranked n-th in the table above. When principals offer a contract profile that is ranked n-th at Stage 1, we let the agent send messages that induce the n-th ranked subsets of $Y_{j_1} \times Y_{j_2}$ at Stage 2. As a result, the agent's incentive compatibility holds at Stage 2.

j_2's incentive compatibility also holds because she achieves the maximal utility on the equilibrium path. Finally, suppose j_1 unilaterally deviates from the equilibrium. Since j_2's equilibrium contract leads only to $\{0,1\}$, by the table above, the contract profile induced by j_1's deviation can be ranked only lower, i.e., not a profitable deviation for j_1.

However, the equilibrium allocation described above cannot be replicated by menu contracts. We prove this by contradiction. Suppose that we can use menu contracts to replicate it. On the equilibrium path, the principals choose $(y_{j_1} = 1, y_{j_2} = 1)$. Thus, principal j_2's equilibrium menu contract must contain 1. Then, principal j_1 finds it profitable to deviate to the degenerate menu $\{0\}$, which would induce the continuation equilibrium $(y_{j_1} = 0, y_{j_2} = 1)$, i.e., both principal j_1 and the agent achieve the highest utility, 8.

5 Simpler contract spaces than $C^{\mathcal{A}}$

5.1 $\left[C^I, C^{II}\right]$-equilibrium and the goal

Given $\mathcal{A} = \mathcal{A}^{non-delegated}$, we follow the same strategy of Peters (2001) to characterize equilibrium allocations. That is, we will identify two simple contract spaces, C^I and C^{II}, and prove that it suffers no loss of generality for principals to focus on C^I and C^{II}, one for the equilibrium path and the other for off-equilibrium paths. Thus, we first define a new notion of $\left[C^I, C^{II}\right]$-equilibrium as follows.

Definition 2 *Given two generic contract spaces, C^I and C^{II}, in a model $\langle \mathcal{A},\ \Gamma,\ \Psi\rangle$, $(c, s, t) \in C \times S \times T$ is a $\left[C^I, C^{II}\right]$-equilibrium if $c \in C^I$ and (c, s, t) is a $\widehat{C}$-equilibrium, where*

$$\widehat{C} \equiv \times_{k\in\mathcal{J}} \widehat{C}_k \equiv \times_{k\in\mathcal{J}} \left(\{c_k\} \cup C_k^{II}\right).$$

Let $\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^I, C^{II}\right]}$ be the set of all $\left[C^I, C^{II}\right]$-equilibria in the model $\langle \mathcal{A},\ \Gamma,\ \Psi\rangle$, and

$$Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^I, C^{II}\right]}} \equiv \left\{z^{(c,s,t)} \in Z : (c, s, t) \in \mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^I, C^{II}\right]}\right\}.$$

It is straightforward to see $\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}C^{\mathcal{A}}} \equiv \mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}$. Thus, for each model $\left\langle \mathcal{A}^{non-delegated},\ \Gamma,\ \Psi\right\rangle$, we aim to find the simple contract spaces, C^I and C^{II}, such that

$$Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} = Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^I, C^{II}\right]}}.$$

We propose such simple contract spaces in Sections 5.2 and 5.3.

5.2 Menu-of-menu-with-recommendation contracts

Given $j \in \mathcal{J}$, pick any $E_j \in 2^{Y_j} \diagdown \{\varnothing\}$, and we say that $[E_j, y_j]$ is a menu with a recommendation if and only if $y_j \in E_j$. Define

$$M_j^R \equiv \left\{[E_j, y_j] : E_j \in 2^{Y_j} \diagdown \{\varnothing\} \text{ and } y_j \in E_j\right\},$$

i.e., M_j^R is the set of all menus with a recommendation. Let $M^R \equiv \times_{k\in\mathcal{J}} M_k^R$.

Definition 3 *A menu-of-menu-with-recommendation contract for principal j is a function, $c_j : K_j \to 2^{Y_j} \diagdown \{\varnothing\}$ such that $K_j \in 2^{M_j^R} \diagdown \{\varnothing\}$ and $c_j([E_j, y_j]) = E_j$, $\forall [E_j, y_j] \in K_j$.*

When the agent sends a message $[E_j, y_j] \in K_j$, the interpretation is that he chooses the menu of actions E_j along with recommending y_j to principal j. Nonetheless, this recommendation is not binding, and principal j can still choose any action in E_j. For example, suppose that $K_j = \{[\{a, b\}, a], [\{c, d\}, c], [\{e, f, g\}, e], [\{e, f, g\}, f]\}$. Then, a menu-of-menu-with-recommendation contract $c_j : K_j \to 2^{Y_j} \diagdown \{\varnothing\}$ has the following property:

$$c_j([\{a, b\}, a]) = \{a, b\}, \; c_j([\{c, d\}, c]) = \{c, d\},$$
$$c_j([\{e, f, g\}, e]) = c_j[\{e, f, g\}, f] = \{e, f, g\}.$$

Let C_j^R be the set of all possible menu-of-menu-with-recommendation contracts for principal j, $C^R \equiv \times_{k \in \mathcal{J}} C_k^R$ and $C_{-j}^R \equiv \times_{k \neq j} C_k^R$.

5.3 Menu-of-menu-with-full-recommendation contracts

We now define another class of contracts. For any $E_j \in 2^{Y_j} \diagdown \{\varnothing\}$, the following is a menu of E_j with *full recommendation.*

$$\{[E_j, y_j] : y_j \in E_j\}.$$

For example, with $E_j = \{a, b, c\}$, a menu of $\{a, b, c\}$ with full recommendation is

$$\{[\{a, b, c\}, a], \; [\{a, b, c\}, b], \; [\{a, b, c\}, c]\}.$$

Definition 4 *A menu-of-menu-with-full-recommendation contract for principal j is a function, $c_j : L_j \cup H_j \to 2^{Y_j} \diagdown \{\varnothing\}$ with*

$$H_j = \{[E_j, y_j] : y_j \in E_j\} \text{ for some } E_j \in 2^{Y_j} \diagdown \{\varnothing\},$$
$$L_j \subset 2^{Y_j} \diagdown \{E_j\},$$

such that

$$c_j(E_j') = E_j', \; \forall E_j' \in L_j,$$
$$c_j([E_j, y_j]) = E_j, \; \forall [E_j, y_j] \in H_j.$$

Let C_j^F denote the set of all menu-of-menu-with-full-recommendation contracts for principal j, $C^F \equiv \times_{k\in\mathcal{J}} C_k^F$, and $C_{-j}^F \equiv \times_{k\in\mathcal{J}\diagdown\{j\}} C_k^F$. Let M_j^F denote the set of all messages that could possibly be included in the domain of a menu-of-menu-with-full-recommendation contracts for principal j, i.e., $M_j^F = 2^{Y_j} \cup M_j^R$. Let $M^F \equiv \times_{k\in\mathcal{J}} M_k^F$, and $M_{-j}^F \equiv \times_{k\in\mathcal{J}\diagdown\{j\}} M_k^F$.

Let us illustrate the domain of a menu-of-menu-with-full-recommendation contract, $c_j : L_j \cup H_j \rightarrow 2^{Y_j} \diagdown \{\varnothing\}$. The set L_j is a (possibly empty) subset of 2^{Y_j}, and $H_j = \{[E_j, y_j] : y_j \in E_j\}$ is a menu of E_j with full recommendation such that E_j is non-empty and $E_j \notin L_j$. For example,

$$L_j \cup H_j = \{\{a,b\}\,,\ \{c,d,e\}\,,\ [\{f,g,h\}\,,f]\,,\ [\{f,g,h\}\,,g]\,,\ [\{f,g,h\}\,,h]\}\,,$$

and we thus have:

$$\begin{aligned} c_j(\{a,b\}) &= \{a,b\}\,,\ c_j(\{c,d,e\}) = \{c,d,e\}\,, \\ c_j([\{f,g,h\}\,,f]) &= c_j([\{f,g,h\}\,,g]) = c_j([\{f,g,h\}\,,h]) = \{f,g,h\}\,. \end{aligned}$$

The interpretation of c_j is: principal j asks the agent to choose a subset in the menu of menus $\{\{a,b\}\,,\ \{c,d,e\}\,,\ \{f,g,h\}\}$; the agent may choose $\{a,b\}$, or $\{c,d,e\}$, or $\{f,g,h\}$; if and only if the agent chooses $\{f,g,h\}$, the agent must, in addition, recommend an action in $\{f,g,h\}$, i.e., f or g or h. Nevertheless, the recommendation is not binding.

Any menu contract (as defined in Section 4.1), e.g., $\{a,b,c\}$, can be viewed as a menu-of-menu-with-full-recommendation contract because we can set $L_j = \{\{a\}\,,\{b\}\}$ and $H_j = \{[\{c\},c]\}$. Therefore, C^P can be viewed as a strict subset of C^F. Furthermore, given any menu-of-menu-with-full-recommendation contract, $c_j : L_j \cup H_j \rightarrow 2^{Y_j} \diagdown \{\varnothing\}$, since a recommendation is non-binding, for each $D_j \in L_j$, we can add an arbitrary recommendation $y_j \in D_j$. Thus, any menu-of-menu-with-full-recommendation contract can be viewed as a menu-of-menu-with-recommendation contract, i.e.,

$$C^P \subsetneq C^F \subsetneq C^R. \tag{15}$$

6 Full Equilibrium characterization I

Given $\mathcal{A} = \mathcal{A}^{non-delegated}$, we focus on three models in this section: $\langle \Gamma^{private},\ \Psi^{private} \rangle$, $\langle\ \Gamma^{public},\ \Psi^{private} \rangle$ and $\langle\ \Gamma^{public},\ \Psi^{public} \rangle$. We consider the model of $\langle \Gamma^{private},\ \Psi^{public} \rangle$ in the next section because it requires a different full characterization.

In the three models, there is no loss of generality to focus on $\left[C^R, C^F\right]$-equilibria.

Theorem 2 *Suppose $\mathcal{A} = \mathcal{A}^{non-delegated}$. We have*

$$Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle \text{-} \left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} = Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle \text{-} \left[C^{R}, C^{F}\right]}}, \tag{16}$$

$$\forall \langle \Gamma,\ \Psi \rangle \in \left\{ \left\langle \Gamma^{private}, \Psi^{private} \right\rangle,\ \left\langle \Gamma^{public}, \Psi^{private} \right\rangle,\ \left\langle \Gamma^{public}, \Psi^{public} \right\rangle \right\}.$$

To prove this theorem, we need two technical results.

Proposition 1 *Suppose $\mathcal{A} = \mathcal{A}^{non-delegated}$. We have*

$$Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle \text{-} \left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} = Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle \text{-} \left[C^{R}, C^{\mathcal{A}}\right]}},$$

$$\forall \langle \Gamma,\ \Psi \rangle \in \left\{ \left\langle \Gamma^{private}, \Psi^{private} \right\rangle,\ \left\langle \Gamma^{public}, \Psi^{private} \right\rangle,\ \left\langle \Gamma^{public}, \Psi^{public} \right\rangle \right\}.$$

Proposition 2 *Suppose $\mathcal{A} = \mathcal{A}^{non-delegated}$. Fix any $I \in \{\mathcal{A}, P, F, R\}$. We have*

$$Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle \text{-} \left[C^{I}, C^{\mathcal{A}}\right]}} = Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle \text{-} \left[C^{I}, C^{F}\right]}},$$

$$\forall \langle \Gamma,\ \Psi \rangle \in \left\{ \left\langle \Gamma^{private}, \Psi^{private} \right\rangle,\ \left\langle \Gamma^{public}, \Psi^{private} \right\rangle,\ \left\langle \Gamma^{public}, \Psi^{public} \right\rangle \right\}.$$

Proposition 1 says that it suffers no loss generality for principals to offer contracts in C^R on the equilibrium path, and Proposition 2 says that it suffers no loss generality for principals to offer contracts in C^F off the equilibrium paths. The proofs of the two propositions are relegated to Appendix A.2 and A.3.

Proof of Theorem 2. Suppose $\mathcal{A} = \mathcal{A}^{non-delegated}$. Consider any

$$\langle \Gamma,\ \Psi \rangle \in \left\{ \left\langle \Gamma^{private}, \Psi^{private} \right\rangle,\ \left\langle \Gamma^{public}, \Psi^{private} \right\rangle,\ \left\langle \Gamma^{public}, \Psi^{public} \right\rangle \right\}.$$

Proposition 1 implies

$$Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle \text{-} \left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} = Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle \text{-} \left[C^{R}, C^{\mathcal{A}}\right]}}, \tag{17}$$

and Proposition 2 implies

$$Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle \text{-} \left[C^{R}, C^{\mathcal{A}}\right]}} = Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle \text{-} \left[C^{R}, C^{F}\right]}}. \tag{18}$$

Finally, (17) and (18) imply (16). ■

In the model of $\left\langle \mathcal{A}^{non-delegated},\ \Gamma^{private},\ \Psi^{private} \right\rangle$, the full characterization can be sharpened as follows. The proof of Theorem 3 is relegated to Appendix A.4.

Theorem 3 *Suppose $\langle \mathcal{A},\ \Gamma,\ \Psi \rangle = \left\langle \mathcal{A}^{non-delegated},\ \Gamma^{private},\ \Psi^{private} \right\rangle$. We have*

$$Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle \text{-} \left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} = Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle \text{-} \left[C^{P}, C^{F}\right]}}. \tag{19}$$

Three remarks on Theorem 3 are in order. First, given $\left\langle \mathcal{A}^{non-delegated},\ \Gamma^{private},\ \Psi^{private}\right\rangle$, Theorem 3 says that it suffers no loss of generality for principals to offer menu contracts on the equilibrium path, which sharpens Theorem 2, due to $C^P \subset C^R$ (in (15)). In this sense, the menu theorem holds partially (i.e., on the equilibrium path), but the example in Section 2 shows that it suffers loss of generality for principals to offer menu contracts on off-equilibrium paths. Second, Theorem 3 applies only to $\left\langle \mathcal{A}^{non-delegated},\ \Gamma^{private},\ \Psi^{private}\right\rangle$, but not to the other three models as shown by the examples in Sections 4.2 and 7.1. Third, with $\mathcal{A} = \mathcal{A}^{delegated}$, the menu theorem (i.e., Theorem 1) says

$$Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} = Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{P}, C^{P}\right]}},$$

i.e., $Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{P}, C^{P}\right]}}$ is both an upper bound and a lower bound for $Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}}$. Given $\left\langle \mathcal{A}^{non-delegated},\ \Gamma^{private},\ \Psi^{private}\right\rangle$, Theorem 4 shows that $Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{P}, C^{P}\right]}}$ remains an upper bound, but the example in Section 2 shows that it is no longer a lower bound. This establishes a second sense that the menu theorem holds partially given $\left\langle \mathcal{A}^{non-delegated},\ \Gamma^{private},\ \Psi^{private}\right\rangle$. The proof of Theorem 4 is relegated to Appendix A.5.

Theorem 4 *In the model* $\langle \mathcal{A},\ \Gamma,\ \Psi\rangle = \left\langle \mathcal{A}^{non-delegated},\ \Gamma^{private},\ \Psi^{private}\right\rangle$*, we have*

$$Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} \subset Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{P}, C^{P}\right]}}.$$

7 Full equilibrium Characterization II

Throughout this section, we focus on $\left\langle \mathcal{A}^{non-delegated},\ \Gamma^{private},\ \Psi^{public}\right\rangle$. Theorem 2 shows

$$Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} = Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{R}, C^{F}\right]}} \tag{20}$$

for the other three models. Thus, it is natural to conjecture that (20) still holds under $\left\langle \mathcal{A}^{non-delegated},\ \Gamma^{private},\ \Psi^{public}\right\rangle$. However, we use an example to disprove this in Section 7.1. Furthermore, we provide a full characterization of $Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}}$ in Section 7.2.

7.1 A counterexample

To disprove (20), suppose that

$$\Theta = \left\{\theta^1, \theta^4\right\}, \mathcal{J} = \{j_1, j_2\}, Y_{j_1} = Y_{j_2} = \{1, 2, 3, 4\}.$$

The common prior is $p\left(\theta^1\right)=p\left(\theta^4\right)=1/2$. The preference is defined as follows.

$$v_{j_1}\left[\left(y_{j_1},y_{j_2}\right),\theta^1\right]=v_{j_2}\left[\left(y_{j_1},y_{j_2}\right),\theta^1\right]=\begin{cases}8, & \text{if } \left(y_{j_1},y_{j_2}\right)=(1,1);\\ 9, & \text{if } \left(y_{j_1},y_{j_2}\right)\in\{(3,1),(4,1)\};\\ y_{j_1}-y_{j_2}+4, & \text{if } \{y_{j_1},y_{j_2}\}\subset\{1,2\} \text{ or } \{y_{j_1},y_{j_2}\}\subset\{3,4\};\\ y_{j_2}-y_{j_1}, & \text{otherwise.}\end{cases} \tag{21}$$

$$v_{j_1}\left[\left(y_{j_1},y_{j_2}\right),\theta^4\right]=v_{j_2}\left[\left(y_{j_1},y_{j_2}\right),\theta^4\right]=\begin{cases}8, & \text{if } \left(y_{j_1},y_{j_2}\right)=(4,4);\\ 9, & \text{if } \left(y_{j_1},y_{j_2}\right)\in\{(4,1),(4,2)\};\\ y_{j_1}-y_{j_2}+4, & \text{if } \{y_{j_1},y_{j_2}\}\subset\{1,2\} \text{ or } \{y_{j_1},y_{j_2}\}\subset\{3,4\};\\ y_{j_2}-y_{j_1}, & \text{otherwise.}\end{cases} \tag{22}$$

$$u\left[\left(y_{j_1},y_{j_2}\right),\theta^1\right]=u\left[\left(y_{j_1},y_{j_2}\right),\theta^4\right]=\begin{cases}1, & \text{if } \left(y_{j_1},y_{j_2}\right)=(1,4);\\ 0, & \text{otherwise}\end{cases} \tag{23}$$

For each $j\in\mathcal{J}$, fix three distinct messages, m_j^1, m_j^2 and m_j^4. Define a contract, $c_j^*: M_j^{\mathcal{A}}\to 2^{Y_j}$:

$$c_j^*\left(m_j\right)=\begin{cases}\{1,2\}, & \text{if } m_j=m_j^1,\\ \{3,4\}, & \text{if } m_j=m_j^4,\\ \{2\}, & \text{if } m_j=m_j^2,\\ \{2\}, & \text{otherwise}\end{cases}.$$

Consider the following equilibrium.

$$\text{on the equilibrium path:}\begin{pmatrix}\text{principal } j_1 \text{ offers } c_{j_1}^*,\\ \text{principal } j_2 \text{ offers } c_{j_2}^*,\end{pmatrix}$$

$$\text{on the equilibrium path:}\begin{pmatrix}\text{at state } \theta^1: \text{ the agent sends } m_{j_1}^1 \text{ to } j_1 \text{ and } m_{j_2}^1 \text{ to } j_2,\\ \text{at state } \theta^4: \text{ the agent sends } m_{j_1}^4 \text{ to } j_1 \text{ and } m_{j_2}^4 \text{ to } j_2,\end{pmatrix}$$

$$\text{on the equilibrium path:}\begin{pmatrix}\text{principals chooses } (1,1), \text{ upon receiving } \left(m_{j_1}^1,m_{j_2}^1\right),\\ \text{principals chooses } (4,4), \text{ upon receiving } \left(m_{j_1}^4,m_{j_2}^4\right).\end{pmatrix}$$

On the equilibrium path, the induced outcome is $(1,1)$ at state θ^1 and $(4,4)$ at state θ^4. To sustain this as an equilibrium, the agent and principals take the following (behavior) strategies at Stages 2 and 3.

At stage 2, off the equilibrium path

$$\text{the agent takes } s \equiv [s_k : C^{\mathcal{A}} \times \Theta \to M_k^{\mathcal{A}}]_{k \in \mathcal{J}} \text{ such that}$$

$$c\left[s\left(c,\theta\right)\right] \neq \{1\} \times \{4\} \Rightarrow \left(\begin{array}{c} \nexists m \in M^{\mathcal{A}} \text{ such that } c\left(m\right) = \{1\} \times \{4\}, \\ \text{and } s\left(c,\theta\right) = \left(m_{j_1}^2, m_{j_2}^2\right) \end{array} \right),$$

$$\forall \left(c,\theta\right) \in \left[C^{\mathcal{A}} \diagdown \left\{\left(c_{j_1}^*, c_{j_2}^*\right)\right\}\right] \times \Theta.$$

That is, we consider two cases: (1) if there exists $m \in M^{\mathcal{A}}$ such that $c\left(m\right) = \{1\} \times \{4\}$, we let the agent send m, which achieves the maximal utility for the agent; (2) otherwise, the agent always sends $\left(m_{j_1}^2, m_{j_2}^2\right)$. If there exists no $m \in M^{\mathcal{A}}$ such that $c\left(m\right) = \{1\} \times \{4\}$, as will be clear, principals will never play $(1,4)$ at Stage 3, regardless the agent's messages at Stage 2. By (23), the agent's incentive compatibility holds.

At Stage 3, let each principal j adopt the (behavior) strategy

$$t_j : C_j^{\mathcal{A}} \times M_j^{\mathcal{A}} \times M_j^{\mathcal{A}} \to Y_j,$$

and the corresponding beliefs described as follows. First, fix any $j \in \mathcal{J}$, and suppose j unilaterally deviates to $c_j \neq c_j^*$, and the agent sends $\left(m_{j_1}^2, m_{j_2}^2\right)$ at Stage 2. Then, principal j cannot confirm deviation from the agent or the other principal. Thus, j must believe that the other principal offers c_{-j}^* at Stage 1, and there is equal probability for θ^1 and θ^4. Given c_{-j}^*, the message m_{-j}^2 pins down the action of 2 for principal $-j$ at Stage 3. Thus, let principal j take the following action at Stage 3.

$$t_j\left[c_j, \left(m_{j_1}^2, m_{j_2}^2\right)\right] \in \arg \max_{y_j \in c_j\left(m_j^2\right)} \left(\frac{1}{2} v_j \left[\left(y_j, y_{-j} = 2\right), \theta^1\right] + \frac{1}{2} v_j \left[\left(y_j, y_{-j} = 2\right), \theta^4\right] \right).$$

Clearly, in this case, principals' incentive compatibility holds at Stage 3. Furthermore, by (21) and (22)), such a unilateral deviation would induce a payoff less than 8 (i.e., the equilibrium payoff) at both states, i.e., this is not a profitable deviation.

Second, consider all of the other off-equilibrium paths. For each $j \in \mathcal{J}$ and each $y_j \in Y_j$, let $c_j^{y_j} : M_j^{\mathcal{A}} \to 2^{Y_j}$ denote the following degenerate contract.

$$c_j^{y_j}\left(m_j\right) = \{y_j\}, \forall m_j \in M_j^{\mathcal{A}}.$$

Principal j_1 takes $t_{j_1} : C_{j_1}^{\mathcal{A}} \times M_{j_1}^{\mathcal{A}} \times M_{j_2}^{\mathcal{A}} \to Y_{j_1}$ such that

$$\left(\begin{array}{c} \left(c_{j_1}, m_{j_1}, m_{j_2}\right) \notin \left\{\left(c_{j_1}^*, m_{j_1}^1, m_{j_2}^1\right), \left(c_{j_1}^*, m_{j_1}^4, m_{j_2}^4\right)\right\}, \\ \text{or } \left(c_{j_1}, m_{j_1}, m_{j_2}\right) \notin \left(C_{j_1}^{\mathcal{A}} \diagdown \left\{c_{j_1}^*\right\}\right) \times \left\{\left(m_{j_1}^2, m_{j_2}^2\right)\right\} \end{array} \right) \Rightarrow \left(\begin{array}{c} t_{j_1}\left(c_{j_1}, m_{j_1}, m_{j_2}\right) = \max c_{j_1}\left(m_{j_1}\right) \equiv \widehat{y}, \\ j_1 \text{ believes } \left[\theta = \theta^1 \text{ and } c_{j_2} = c_{j_2}^{\widehat{y}}\right] \\ \text{with probability } 1 \end{array} \right),$$

and Principal j_2 takes $t_{j_2} : C^{\mathcal{A}}_{j_2} \times M^{\mathcal{A}}_{j_1} \times M^{\mathcal{A}}_{j_2} \to Y_{j_2}$ such that

$$\left(\begin{array}{c} (c_{j_2}, m_{j_1}, m_{j_2}) \notin \left\{ \left(c^*_{j_2}, m^1_{j_1}, m^1_{j_2}\right), \left(c^*_{j_2}, m^4_{j_1}, m^4_{j_2}\right) \right\}, \\ \text{or } (c_{j_2}, m_{j_1}, m_{j_2}) \notin \left(C^{\mathcal{A}}_{j_1} \diagdown \left\{c^*_{j_1}\right\}\right) \times \left\{ \left(m^2_{j_1}, m^2_{j_2}\right) \right\} \end{array} \right) \Rightarrow \left(\begin{array}{c} t_{j_2}(c_{j_2}, m_{j_1}, m_{j_2}) = \min c_{j_2}(m_{j_2}) \equiv \widetilde{y}, \\ j_2 \text{ believes } \left[\theta = \theta^4 \text{ and } c_{j_1} = c^{\widetilde{y}}_{j_1}\right] \\ \text{with probability } 1 \end{array} \right).$$

It is easy to check that principals' incentive compatibility at Stage 3 holds.

However, this equilibrium allocation cannot be replicated as an equilibrium allocation in the $\left[C^R, C^F\right]$ game. We prove by contradiction. Suppose we can achieve the allocation with a $\left[C^R, C^F\right]$-equilibrium. Let us focus on the equilibrium path. At State θ^1, the agent chooses $E^{\theta^1}_{j_1}$ for j_1 and $E^{\theta^1}_{j_2}$ for j_2, and at State θ^4, the agent chooses $E^{\theta^4}_{j_1}$ for j_1 and $E^{\theta^4}_{j_2}$ for j_2. Since it implements $(1,1)$ at θ^1 and $(4,4)$ at θ^4, we have

$$1 \in E^{\theta^1}_{j_1} \text{ and } 4 \in E^{\theta^4}_{j_2}.$$

At state θ^1, on the equilibrium path, principal j_1 expects that j_2 would choose 1 at Stage 3. By the preference of j_1 (i.e., (21)), we have

$$\{3,4\} \cap E^{\theta^1}_{j_1} = \varnothing.$$

Similarly, at state θ^4, on the equilibrium path, principal j_2 expects that j_1 would choose 4 at Stage 3. By the preference of j_1 (i.e., (22)), we have

$$\{1,2\} \cap E^{\theta^4}_{j_2} = \varnothing.$$

That is,

$$1 \in E^{\theta^1}_{j_1} \subset \{1,2\} \text{ and } 4 \in E^{\theta^4}_{j_2} \subset \{3,4\}.$$

Then, at State θ^1 (also at θ^4), the agent finds it profitable to deviate to choose $\left(E^{\theta^1}_{j_1}, E^{\theta^4}_{j_2}\right)$ for the principals. Upon observing this, the dominant strategy for principal j_1 is to choose 1 and the dominant strategy for principal j_2 is to choose 4. That is, the principals chooses $(1,4)$, which is strictly better than the equilibrium allocation $(1,1)$ for the agent.

The key point is as follows. When principal j offers a contract in $C^{\mathcal{A}}_j$, the other principals may not correctly infer the subset of actions available for j conditional on the agent's message to j under private announcement and public communication. However, if principal j is restricted to offer a contract in C^R_j or C^F_j, the agent's message to j alone fully reveals the subset of actions available for j under private announcement and public communication. This can create a profitable deviation for the agent at Stage 2 on the equilibrium path such that it induces a unique profile of equilibrium action choices by principals at Stage 3, which makes the agent strictly better off, preventing the intended equilibrium allocation from happening.

7.2 A full characterization

Nevertheless, the menu-of-menu-with-recommendation theorem can be easily adapted in $\left\langle \mathcal{A}^{non-delegated}, \Gamma^{private}, \Psi^{public} \right\rangle$. For each $j \in \mathcal{J}$, let M_j^{R-F} be the set of messages that are used contracts in $C_j^R \cup C_j^F$. For each $y_j \in Y_j$, let $c_j^{y_j} : M_j^{R-F} \rightarrow 2^{Y_j} \diagdown \{\varnothing\}$ denote the following degenerate contract.

$$c_j^{y_j}(m_j) = \{y_j\}, \forall m_j \in M_j^{R-F}. \tag{24}$$

Define

$$C^{F^*} \equiv \times_{j \in \mathcal{J}} C_j^{F^*} \equiv \times_{j \in \mathcal{J}} \left[C_j^F \cup \left\{ c_j^{y_j} : y_j \in Y_j \right\} \right].$$

Note that degenerate contracts are designed in a way so that adding such contracts prevents principals from correctly inferring the subset of actions available for each principal j conditional on the agent's message to j.

Theorem 5 *Given* $\langle \mathcal{A}, \Gamma, \Psi \rangle = \left\langle \mathcal{A}^{non-delegated}, \Gamma^{private}, \Psi^{public} \right\rangle$, *we have*

$$Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle - \left[C^{\mathcal{A}}, C^{\mathcal{A}} \right]}} = Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle - \left[C^{R}, C^{F^*} \right]}}.$$

The proof of Theorem 5 is relegated to Han and Xiong (2022).

8 Common agency with imperfect commitment

In this section, we adapt our model to describe imperfect commitment *à la* Bester and Strausz (2000, 2001, 2007). In order to achieve this, we just need to make three changes to the model in Section 3. First, for each $j \in \mathcal{J}$, assume $Y_j = Y_j^1 \times Y_j^2$. Second, for each $j \in \mathcal{J}$, there is an exogenous function $\phi_j : Y_j^1 \longrightarrow 2^{Y_j^2} \diagdown \{\varnothing\}$. Third, a contract of principal j is $c_j : M_j^{\mathcal{A}} \longrightarrow Y_j^1$. Let $\widetilde{C}_j^{\mathcal{A}} \equiv \left(Y_j^1\right)^{M_j^{\mathcal{A}}}$ denote j's contract space, and $\widetilde{C}^{\mathcal{A}} = \left(\widetilde{C}_j^{\mathcal{A}}\right)_{j \in \mathcal{J}}$. In this game, we follow the same timeline in Section 3.2.1: each principal j simultaneously offers $c_j \in \widetilde{C}_j^{\mathcal{A}}$ at Stage 1; the agent sends $m_j \in M_j^{\mathcal{A}}$ to each principal j at Stage 2, which pins down the action $c_j(m_j) \in Y_j^1$ and the subset $\phi_j[c_j(m_j)] \in 2^{Y_j^2} \diagdown \{\varnothing\}$ for principal j; at Stage 3, each principal j simultaneously takes an action $\left(c_j(m_j), \ y_j^2 \in \phi_j[c_j(m_j)]\right)$.

In fact, the model in Section 3 can be viewed as a special case of the model with imperfect commitment. Though Y_j in our model in Section 3 may not be directly decomposed to contractible and non-contractible components, we can define a new action space for principal

j as follows.

$$\widetilde{Y}_j^1 = 2^{Y_j} \diagdown \{\varnothing\}, \widetilde{Y}_j^2 = Y_j, \widetilde{Y}_j = \widetilde{Y}_j^1 \times \widetilde{Y}_j^2,$$
$$\widetilde{\phi}_j\left(y_j^1\right) = y_j^1, \forall y_j^1 \in \widetilde{Y}_j^1.$$

Thus, the common-agency-with-imperfect-commitment model[17] with $\left(\widetilde{Y}_j, \widetilde{\phi}_j\right)_{j\in\mathcal{J}}$ is equivalent to the model in Section 3.[18]

Theorem 2 can be easily extended to a common-agency-with-imperfect-commitment model with $\left(Y_j = Y_j^1 \times Y_j^2,\ \phi_j\right)_{j\in\mathcal{J}}$. Consider $\Xi_j \equiv \left\{\left(y_j^1, y_j^2\right) : y_j^1 \in Y_j^1 \text{ and } y_j^2 \in \phi_j\left(y_j^1\right)\right\}$. In this setup, a menu-of-menu-with-recommendation contract is represented by a non-empty subset $E_j \in 2^{\Xi_j} \diagdown \{\varnothing\}$, or more precisely, the contract, $c_j : E_j \longrightarrow Y_j^1$ such that $c_j\left(y_j^1, y_j^2\right) \equiv y_j^1$. The interpretation is that, the message $\left(y_j^1, y_j^2\right)$ pins down $y_j^1 \in Y_j^1$ and $\phi_j\left(y_j^1\right) \subset Y_j^2$, and the agent recommends $y_j^2 \in \phi_j\left(y_j^1\right)$ for Stage 3. Let $\widetilde{C}_j^R$ denote the set of j's menu-of-menu-with recommendation contracts, and $\widetilde{C}^R \equiv \left(\widetilde{C}_j^R\right)_{j\in\mathcal{J}}$.

Furthermore, consider $\Sigma_j \equiv \left\{\left\{\left(y_j^1, y_j^2\right) : y_j^2 \in \phi_j\left(y_j^1\right)\right\} : \quad y_j^1 \in Y_j^1\right\}$. A menu-of-menu-with-full-recommendation contract is represented by a subset $L_j \in 2^{Y_j^1}$ and an element $H_j \in \Sigma_j$, or more precisely, the contract, $c_j : L_j \cup H_j \longrightarrow Y_j^1$ such that

$$c_j\left(y_j^1\right) = y_j^1, \forall y_j^1 \in L_j \text{ and } c_j\left(y_j^1, y_j^2\right) = y_j^1, \forall\left(y_j^1, y_j^2\right) \in H_j.$$

Let $\widetilde{C}_j^F$ denote the set of j's menu-of-menu-with-full-recommendation contracts, and $\widetilde{C}^F \equiv$

[17] In this model, principals' and the agent's utility depends only on $\left(\widetilde{Y}_j^2\right)_{j\in\mathcal{J}}$ but not on $\left(\widetilde{Y}_j^1\right)_{j\in\mathcal{J}}$.

[18] One superficial difference between the model in Section 3 and the model with imperfect commitment is that the degree of principals' commitment is endogenous in the former, but seems exogenous in the latter (as described by the exogenous ϕ_j). In fact, ϕ_j can accommodate endogenous commitment. To see this, take $Y_j = Y_j^1 \times Y_j^2$ as principal j's underlying action space. Conditional on choosing y_j^1, let $\digamma_j\left(y_j^1\right) \subset 2^{Y_j^2} \diagdown \{\varnothing\}$ denote the set of subsets of Y_j^2 to which principal j can commit for Stage 3. If $\digamma_j\left(y_j^1\right) = 2^{Y_j^2} \diagdown \{\varnothing\}$ for every $y_j^1 \in Y_j^1$, principal j's commitment is fully endogenous, and if $\left|\digamma_j\left(y_j^1\right)\right| = 1$ for every $y_j^1 \in Y_j^1$, principal j's commitment is fully exogenous. ϕ_j can describe any $\digamma_j$ by changing the action space as follows.

$$\widehat{Y}_j^1 = \left\{\left(y_j^1, P_j^2\right) : y_j^1 \in Y_j^1 \text{ and } P_j^2 \in \digamma_j\left(y_j^1\right)\right\}, \widehat{Y}_j = \widehat{Y}_j^1 \times Y_j^2,$$
$$\widehat{\phi}_j\left(y_j^1, P_j^2\right) = P_j^2, \forall\left(y_j^1, P_j^2\right) \in \widehat{Y}_j^1.$$

Thus, by picking different $\digamma_j$, the common-agency-with-imperect-commitment model with $\left(\widehat{Y}_j, \widehat{\phi}_j\right)_{j\in\mathcal{J}}$ can describe fully exogenous commitment, or fully endogenous commitment, or any intermediate ones.

$\left(\widetilde{C}_j^F\right)_{j\in\mathcal{J}}$. Then, following a similar argument, it is straightforward to prove

$$Z^{\mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[\widetilde{C}^{\mathcal{A}},\widetilde{C}^{\mathcal{A}}\right]}} = Z^{\mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[\widetilde{C}^{R},\widetilde{C}^{F}\right]}},$$

$$\forall\left\langle\Gamma,\ \Psi\right\rangle \in \left\{\left\langle\Gamma^{private},\Psi^{private}\right\rangle,\ \left\langle\Gamma^{public},\Psi^{private}\right\rangle,\ \left\langle\Gamma^{public},\Psi^{public}\right\rangle\right\}.$$

9 Conclusion

The revelation principle is a pillar in mechanism design, and a fundamental question is: what are the indispensable assumptions on which the revelation principle hinges? This classical question has been answered by various papers, e.g., Myerson (1979), McAfee (1993), Bester and Strausz (2001), Pavan, Segal, and Toikka (2014), Doval and Skreta (2021).

In common-agency models, the menu theorem is the counterpart of the revelation principle. What are the indispensable assumptions on which the menu theorem hinges? To the best of our knowledge, this paper is the first one to study this question. Not only do we identify two indispensable assumptions (i.e., delegated contracts and perfect commitment), we also show how the menu theorem should be modified when such assumptions fail.

A Proofs

A.1 Extended contracts

For each $j \in \mathcal{J}$, we say a contract $c_j' : M_j' \longrightarrow 2^{Y_j}\diagdown\{\varnothing\}$ is an extension of another $c_j'' : M_j'' \longrightarrow 2^{Y_j}\diagdown\{\varnothing\}$ (denoted by $c_j' \geq c_j''$) if and only if there exists a *surjective* function $\iota_j : M_j' \longrightarrow M_j''$ such that

$$c_j'(m_j) = c_j''(\iota_j(m_j)), \forall m_j \in M_j',$$

Based on "$\geq$," we define two binary relations. First, for any $I, II \in \{\mathcal{A}, P, F, R\}$, define

$$C^I \sqsupseteq^* C^{II} \Longleftrightarrow \begin{pmatrix} \forall j \in \mathcal{J}, \forall c_j'' \in C_j^{II}, \\ \exists c_j' \in C_j^I, c_j' \geq c_j'' \end{pmatrix}.$$

Clearly, $C^{\mathcal{A}} \sqsupseteq^* C^R \sqsupseteq^* C^F \sqsupseteq^* C^P$. Second, for any $I, II \in \{\mathcal{A}, P, F, R\}$, define

$$C^I \sqsupseteq^{**} C^{II} \Longleftrightarrow \begin{pmatrix} \forall j \in \mathcal{J}, \forall c_j' \in C_j^I, \\ \exists c_j'' \in C_j^{II}, c_j' \geq c_j'' \end{pmatrix}.$$

Clearly, $C^{\mathcal{A}} \sqsupset^{**} C^F$ and $C^P \sqsupset^{**} C^F$, but "$C^I \sqsupset^{**} C^P$" fails for each $I \in \{\mathcal{A}, F, R\}$.

Lemma 1 *For any $I, II, III \in \{\mathcal{A}, P, F, F^*, R\}$, we have*

$$C^I \sqsupset^* C^{III} \Longrightarrow Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^I, C^{II}\right]}} \supset Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{III}, C^{II}\right]}},$$
$$\forall \langle \Gamma,\ \Psi\rangle \in \left\{\Gamma^{private},\ \Gamma^{public}\right\} \times \left\{\Psi^{private},\ \Psi^{public}\right\}.$$

Lemma 2 *For any $I, II, IV \in \{\mathcal{A}, P, F, R\}$, we have*

$$C^{II} \sqsupset^{**} C^{IV} \Longrightarrow Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^I, C^{II}\right]}} \supset Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^I, C^{IV}\right]}},$$
$$\forall \langle \Gamma,\ \Psi\rangle \in \left\{\left\langle \Gamma^{private}, \Psi^{private}\right\rangle,\ \left\langle \Gamma^{public}, \Psi^{private}\right\rangle,\ \left\langle \Gamma^{public}, \Psi^{public}\right\rangle\right\}.$$

A.1.1 Proof of Lemma 1

Fix any $I, II, III \in \{\mathcal{A}, P, F, R\}$. Let M^I, M^{II} and M^{III} denote the message spaces for C^I, C^{II} and C^{III}, respectively. Fix any $(c, s, t) \in \mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{III}, C^{II}\right]}$. We aim to construct a $\left[C^I, C^{II}\right]$-equilibrium that induces the allocation $z^{(c,s,t)}$.

Since $C^I \sqsupset^* C^{III}$, there exits $c^{(c,s,t)} \in C^I$ such that $c_j^{(c,s,t)} \geq c_j$ for every $j \in \mathcal{J}$, and as a result, there exists a surjective function $\iota_j : M_j^I \longrightarrow M_j^{III}$ such that

$$c_j^{(c,s,t)}(m_j) = c_j(\iota_j(m_j)),\ \forall m_j \in M_j^I.$$

We will replicate (c, s, t) with $\left(c^{(c,s,t)}, s^{(c,s,t)}, t^{(c,s,t)}\right) \in \mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^I, C^{II}\right]}$ in two steps. First, with abuse of notation, let $\iota_j^{-1} : M_j^{III} \longrightarrow M_j^I$ denote any injective function such that

$$c_j^{(c,s,t)}\left(\iota_j^{-1}(m_j)\right) = c_j(m_j),\ \forall m_j \in M_j^{III}. \tag{25}$$

Consider the contract $\widehat{c_j^{(c,s,t)}} : \iota_j^{-1}\left(M_j^{III}\right) \longrightarrow 2^{Y_j} \diagdown \{\varnothing\}$ with the restricted domain $\iota_j^{-1}\left(M_j^{III}\right)$:

$$\widehat{c_j^{(c,s,t)}}(m_j) = c_j^{(c,s,t)}(m_j),\ \forall m_j \in \iota_j^{-1}\left(M_j^{III}\right),$$

which, together with (25), implies

$$\widehat{c_j^{(c,s,t)}}(m_j) = c_j(\iota_j(m_j)),\ \forall m_j \in \iota_j^{-1}\left(M_j^{III}\right),$$
$$\text{or equivalently, } \widehat{c_j^{(c,s,t)}}\left(\iota_j^{-1}(m_j)\right) = c_j(m_j),\ \forall m_j \in M_j^{III},$$

That is, each $\widehat{c_j^{(c,s,t)}}$ is the same as c_j, where each message $m_j \in M_j^{III}$ for the latter is translated to $\iota_j^{-1}(m_j) \in \iota_j^{-1}\left(M_j^{III}\right)$ for the former. Denote $\widehat{c^{(c,s,t)}} \equiv \left(\widehat{c_j^{(c,s,t)}}\right)_{j\in\mathcal{J}}$. We will construct an equilibrium $\left(\widehat{c^{(c,s,t)}}, s^{(c,s,t)}, \widehat{t^{(c,s,t)}}\right)$. However, $\widehat{c^{(c,s,t)}} \notin C^I$, and in the second step below, we extend $\left(\widehat{c^{(c,s,t)}}, s^{(c,s,t)}, \widehat{t^{(c,s,t)}}\right)$ to $\left(c^{(c,s,t)}, s^{(c,s,t)}, t^{(c,s,t)}\right) \in \mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi\rangle\text{-}\left[C^I, C^{II}\right]}$.

Replicating s with $s^{(c,s,t)}$:

For each $j \in \mathcal{J}$, define the bijection, $\eta_j : \left\{\widehat{c_j^{(c,s,t)}}\right\} \cup C_j^{II} \longrightarrow \{c_j\} \cup C_j^{II}$ as follows.

$$\eta_j\left(c'_j\right) \equiv \begin{cases} c_j, & \text{if } c'_j = \widehat{c_j^{(c,s,t)}}; \\ c'_j, & \text{if } c'_j \in C_j^{II} \diagdown \left\{\widehat{c_j^{(c,s,t)}}\right\}, \end{cases}$$

i.e., we identify $\widehat{c_j^{(c,s,t)}}$ with c_j. Define

$$s_j^{(c,s,t)}\left(\left(c'_k\right)_{k\in\mathcal{J}}, \theta\right) \equiv \begin{cases} \iota_j^{-1}\left(s_j\left[\left(\eta_k\left(c'_k\right)\right)_{k\in\mathcal{J}},\ \theta\right]\right), & \text{if } c'_j = \widehat{c_j^{(c,s,t)}}; \\ s_j\left[\left(\eta_k\left(c'_k\right)\right)_{k\in\mathcal{J}},\ \theta\right], & \text{if } c'_j \in C_j^{II} \diagdown \left\{\widehat{c_j^{(c,s,t)}}\right\}. \end{cases}$$

That is, $s_j^{(c,s,t)}$ replicates s_j: the agent identifies $\left(c'_k\right)_{k\in\mathcal{J}}$ with $\left(\eta_k\left(c'_k\right)\right)_{k\in\mathcal{J}}$, and follow s_j to send the message $s_j\left[\left(\eta_k\left(c'_k\right)\right)_{k\in\mathcal{J}},\ \theta\right]$ when $c'_j \in C_j^{II} \diagdown \left\{\widehat{c_j^{(c,s,t)}}\right\}$; the message is re-labeled to $\iota_j^{-1}\left(s_j\left[\left(\eta_k\left(c'_k\right)\right)_{k\in\mathcal{J}},\ \theta\right]\right)$ when $c'_j = \widehat{c_j^{(c,s,t)}}$.

Replicating t with $\widehat{t^{(c,s,t)}}$:

For each $\left(c'_k\right)_{k\in\mathcal{J}}$ and each $j \in \mathcal{J}$, define $\xi_j^{\left(c'_k\right)_{k\in\mathcal{J}}} : M_j^I \cup M_j^{II} \longrightarrow M_j^{III} \cup M_j^{II}$:

$$\xi_j^{\left(c'_k\right)_{k\in\mathcal{J}}}\left(m_j\right) = \begin{cases} \iota_j\left(m_j\right) & \text{if } c'_j = \widehat{c_j^{(c,s,t)}}; \\ m_j & \text{if } c'_j \in C_j^{II} \diagdown \left\{\widehat{c_j^{(c,s,t)}}\right\}. \end{cases}$$

That is, $\xi_j^{\left(c'_k\right)_{k\in\mathcal{J}}}\left(m_j\right)$ re-label m_j if and only if $c'_j = \widehat{c_j^{(c,s,t)}}$. Denote $\xi^{\left(c'_k\right)_{k\in\mathcal{J}}} \equiv \left(\xi_j^{\left(c'_k\right)_{k\in\mathcal{J}}}\right)_{j\in\mathcal{J}}$.

For each $j \in \mathcal{J}$, define

$$\widehat{t_j^{(c,s,t)}}\left(\Gamma_j\left(\left(c'_k\right)_{k\in\mathcal{J}}\right), \Psi_j\left(m\right)\right) = t_j\left(\Gamma_j\left(\left(\eta_k\left(c'_k\right)\right)_{k\in\mathcal{J}}\right), \Psi_j\left(\xi^{\left(c'_k\right)_{k\in\mathcal{J}}}\left(m\right)\right)\right),$$

i.e., $\widehat{t_j^{(c,s,t)}}$ replicates t_j subject to re-labeling of the messages (by $\xi_j^{\left(c'_k\right)_{k\in\mathcal{J}}}$).

Clearly, $\left(\widehat{c^{(c,s,t)}}, s^{(c,s,t)}, \widehat{t^{(c,s,t)}}\right)$ is the "same" as (c, s, t) subject to re-labeling of the messages. Similarly, we can transform the principals' beliefs at Stage 3 for (c, s, t) to their beliefs, denoted by $\widehat{b^{(c,s,t)}}$, for $\left(\widehat{c^{(c,s,t)}}, s^{(c,s,t)}, \widehat{t^{(c,s,t)}}\right)$ subject to re-labeling of the messages.

Therefore, $\left(\widehat{c^{(c,s,t)}}, s^{(c,s,t)}, \widehat{t^{(c,s,t)}}\right)$ is an equilibrium. The only problem is $\widehat{c^{(c,s,t)}} \notin C^I$. We now extend $\left(\widehat{c^{(c,s,t)}}, s^{(c,s,t)}, \widehat{t^{(c,s,t)}}\right)$ to $\left(c^{(c,s,t)}, s^{(c,s,t)}, t^{(c,s,t)}\right) \in \mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi\rangle\text{-}\left[C^I, C^{II}\right]}$.

Extending $\left(\widehat{t^{(c,s,t)}},\, \widehat{b^{(c,s,t)}}\right)$ to $\left(t^{(c,s,t)},\, b^{(c,s,t)}\right)$:

For each $j \in \mathcal{J}$, define $\varphi_j : \left(\left\{c_j^{(c,s,t)}\right\} \cup C_j^{II}\right) \longrightarrow \left(\left\{\widehat{c_j^{(c,s,t)}}\right\} \cup C_j^{II}\right)$ as

$$\varphi_j\left(c_j'\right) = \begin{cases} \widehat{c_j^{(c,s,t)}} & \text{if } c_j' = c_j^{(c,s,t)}; \\ c_j' & \text{if } c_j' \in C_j^{II} \diagdown \left\{c_j^{(c,s,t)}\right\}. \end{cases}$$

I.e., φ_j re-label c_j' to $\widehat{c_j^{(c,s,t)}}$ if and only if $c_j' = c_j^{(c,s,t)}$.

For each $j \in \mathcal{J}$ and each $(c_k')_{k\in\mathcal{J}}$, define $\pi_j^{\left(c_k'\right)_{k\in\mathcal{J}}} : M_j^I \cup M_j^{II} \longrightarrow M_j^I \cup M_j^{II}$ as

$$\pi_j^{\left(c_k'\right)_{k\in\mathcal{J}}}\left(m_j\right) = \begin{cases} m_j & \text{if } c_j' \in C_j^{II} \diagdown \left\{c_j^{(c,s,t)}\right\}; \\ \iota_j^{-1}\left(\iota_j\left(m_j\right)\right) & \text{if } c_j' = c_j^{(c,s,t)}. \end{cases}$$

That is, $\pi_j^{\left(c_k'\right)_{k\in\mathcal{J}}}\left(m_j\right)$ re-labels m_j to $\iota_j^{-1}\left(\iota_j\left(m_j\right)\right) \in \iota_j^{-1}\left(M_j^{III}\right)$ if and only if $c_j' = c_j^{(c,s,t)}$.[19] Denote $\pi^{\left(c_k'\right)_{k\in\mathcal{J}}} \equiv \left(\pi_j^{\left(c_k'\right)_{k\in\mathcal{J}}}\right)_{j\in\mathcal{J}}$. For each $j \in \mathcal{J}$, define

$$t_j^{(c,s,t)}\left(\Gamma_j\left(\left(c_k'\right)_{k\in\mathcal{J}}\right), \Psi_j\left(m\right)\right) \equiv \widehat{t_j^{(c,s,t)}}\left(\Gamma_j\left(\varphi_k\left(c_k'\right)_{k\in\mathcal{J}}\right), \Psi_j\left(\pi^{\left(c_k'\right)_{k\in\mathcal{J}}}\left(m\right)\right)\right),$$

i.e., $t_j^{(c,s,t)}$ replicates $\widehat{t_j^{(c,s,t)}}$ subject to identifying each $m_j \in M_j^I \diagdown \iota_j^{-1}\left(M_j^{III}\right)$ with $\iota_j^{-1}\left(\iota_j\left(m_j\right)\right)$. Similarly, define beliefs as follows.

$$b_j^{(c,s,t)}\left(\Gamma_j\left(\left(c_k'\right)_{k\in\mathcal{J}}\right), \Psi_j\left(m\right)\right) \equiv \widehat{b_j^{(c,s,t)}}\left(\Gamma_j\left(\varphi_k\left(c_k'\right)_{k\in\mathcal{J}}\right), \Psi_j\left(\pi^{\left(c_k'\right)_{k\in\mathcal{J}}}\left(m\right)\right)\right).$$

Clearly, $\left(c^{(c,s,t)}, s^{(c,s,t)}, t^{(c,s,t)}\right)$ replicates $\left(\widehat{c^{(c,s,t)}}, s^{(c,s,t)}, \widehat{t^{(c,s,t)}}\right)$, and principals and the agent inherit incentive compatibility. Note that, upon receiving $c_j^{(c,s,t)}$ from principal j, the agent does not find it profitable to deviate to sending messages in $M_j^I \diagdown \iota_j^{-1}\left(M_j^{III}\right)$, because sending $m_j \in M_j^I \diagdown \iota_j^{-1}\left(M_j^{III}\right)$ is equivalent to sending $\iota_j^{-1}\left(\iota_j\left(m_j\right)\right) \in \iota_j^{-1}\left(M_j^{III}\right)$, which is not a profitable deviation in the equilibrium $\left(\widehat{c^{(c,s,t)}}, s^{(c,s,t)}, \widehat{t^{(c,s,t)}}\right)$. Finally, all of the argument above works for any $\langle\Gamma,\ \Psi\rangle \in \left\{\Gamma^{private},\ \Gamma^{public}\right\} \times \left\{\Psi^{private},\ \Psi^{public}\right\}$.■

[19] $c_j^{(c,s,t)}\left(m_j\right) = c_j^{(c,s,t)}\left(\iota_j^{-1}\left(\iota_j\left(m_j\right)\right)\right)$ for every $m_j \in M_j^I$, i.e., m_j and $\iota_j^{-1}\left(\iota_j\left(m_j\right)\right)$ pin down the same subset of actions under the contract $c_j^{(c,s,t)}$.

A.1.2 Proof of Lemma 2

Fix any $I, II, IV \in \{\mathcal{A}, P, F, R\}$. Let M^I, M^{II} and M^{IV} denote the message spaces for C^{II}, C^{II} and C^{IV}, respectively. Fix any $(c, s, t) \in \mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi\rangle\text{-}\left[C^I, C^{IV}\right]}$. We aim to construct a $\left[C^I, C^{II}\right]$-equilibrium that induces $z^{(c,s,t)}$.

Since $C^{II} \sqsupset^{**} C^{IV}$, there exists a function $\psi_j : C_j^{II} \longrightarrow C_j^{IV}$ for each $j \in \mathcal{J}$ such that

$$c'_j \geq \psi_j\left(c'_j\right), \forall c'_j \in C_j^{II}.$$

Thus, for each $c'_j \in C_j^{II}$, there exists a surjective $\iota_j^{c'_j} : M_j^{II} \longrightarrow M_j^{IV}$ such that

$$c'_j\left(m_j\right) = \psi_j\left(c'_j\right)\left(\iota_j^{c'_j}\left(m_j\right)\right), \forall m_j \in M_j^{II}.$$

As in Appendix A.1.1, let $\left(\iota_j^{c'_j}\right)^{-1} : M_j^{IV} \longrightarrow M_j^{II}$ denote any injective function such that

$$c'_j\left(\left(\iota_j^{c'_j}\right)^{-1}\left(m_j\right)\right) = \psi_j\left(c'_j\right)\left(m_j\right), \forall m_j \in M_j^{IV}.$$

Consider the contract $\widehat{c'_j} : \left(\iota_j^{c'_j}\right)^{-1}\left(M_j^{IV}\right) \longrightarrow 2^{Y_j} \diagdown \{\varnothing\}$ with the restricted domain:

$$\widehat{c'_j}\left(m_j\right) = c'_j\left(m_j\right), \forall m_j \in \left(\iota_j^{c'_j}\right)^{-1}\left(M_j^{IV}\right),$$

i.e., each $\widehat{c'_j}$ is the same as $\psi_j\left(c'_j\right)$, where each message $m_j \in M_j^{IV}$ for the latter is translated to $\left(\iota_j^{c'_j}\right)^{-1}\left(m_j\right) \in \left(\iota_j^{c'_j}\right)^{-1}\left(M_j^{IV}\right)$ for the former. Denote $\widehat{c'} \equiv \left(\widehat{c'_j}\right)_{j \in \mathcal{J}}$. Define

$$\widehat{C}_j^{II} \equiv \left\{\widehat{c'_j} : c'_j \in C_j^{II}\right\}, \forall j \in \mathcal{J} \text{ and } \widehat{C}^{II} \equiv \times_{j \in \mathcal{J}} \widehat{C}_j^{II}.$$

Clearly, we have $Z^{\mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi\rangle\text{-}\left[C^I, \widehat{C}^{II}\right]}} \supset Z^{\mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi\rangle\text{-}\left[C^I, C^{IV}\right]}}$, i.e., we can replicate any $(c, s, t) \in \mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi\rangle\text{-}\left[C^{III}, C^{II}\right]}$ with some $\left(c, \widehat{s}, \widehat{t}\right) \in \mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi\rangle\text{-}\left[C^I, \widehat{C}^{II}\right]}$, which summarizes the re-labeling of messages (i.e., $\left(\iota_j^{c'_j}\right)^{-1}$) as in Appendix A.1.1.

As in Appendix A.1.1, we can extend each $\widehat{c'_j} \in \widehat{C}_j^{II}$ to $c'_j \in C_j^{II}$. The difference between $\widehat{c'_j}$ and c'_j is that the agent cannot send messages in $M_j^{II} \diagdown \left(\iota_j^{c'_j}\right)^{-1}\left(M_j^{IV}\right)$ under $\widehat{c'_j}$, while he can under c'_j. Suppose principal j identify each $m_j \in M_j^{II} \diagdown \left(\iota_j^{c'_j}\right)^{-1}\left(M_j^{IV}\right)$ with

$$\left(\iota_j^{c'_j}\right)^{-1}\left[\iota_j^{c'_j}\left(m_j\right)\right] \in \left(\iota_j^{c'_j}\right)^{-1}\left(M_j^{IV}\right),$$

and then replicate $\widehat{t}$. Given this, the agent does not have incentive to send any message $m_j \in M_j^{II} \diagdown \left(\iota_j^{c'_j}\right)^{-1}\left(M_j^{IV}\right)$, because sending m_j is equivalent to sending $\left(\iota_j^{c'_j}\right)^{-1}\left[\iota_j^{c'_j}\left(m_j\right)\right] \in \left(\iota_j^{c'_j}\right)^{-1}\left(M_j^{IV}\right)$. Rigorously, Extending $\left(\widehat{t}, \widehat{b}\right)$ to (t^*, b^*):

For each $j \in \mathcal{J}$, define $\varphi_j : \left(\{c_j\} \cup C_j^{II}\right) \longrightarrow \left(\{c_j\} \cup \widehat{C}_j^{II}\right)$,

$$\varphi_j\left(c_j'\right) = \begin{cases} c_j & \text{if } c_j' = c_j; \\ \widehat{c_j'} & \text{if } c_j' \in C_j^{II} \diagdown \{c_j\}. \end{cases}$$

For each $j \in \mathcal{J}$ and each $(c_k')_{k\in\mathcal{J}}$, define $\pi_j^{(c_k')_{k\in\mathcal{J}}} : M_j^I \cup M_j^{II} \longrightarrow M_j^I \cup M_j^{II}$:

$$\pi_j^{(c_k')_{k\in\mathcal{J}}}(m_j) = \begin{cases} m_j & \text{if } c_j' = c_j; \\ \left(\iota_j^{c_j'}\right)^{-1}\left[\iota_j^{c_j'}(m_j)\right] & \text{if } c_j' \in C_j^{II} \diagdown \{c_j\}. \end{cases}$$

For each $j \in \mathcal{J}$, define

$$\begin{aligned} t_j^*\left(\Gamma_j\left((c_k')_{k\in\mathcal{J}}\right), \Psi_j(m)\right) &\equiv \widehat{t_j}\left(\Gamma_j\left(\varphi_k(c_k')_{k\in\mathcal{J}}\right), \Psi_j\left(\pi^{(c_k')_{k\in\mathcal{J}}}(m)\right)\right), \\ b_j^*\left(\Gamma_j\left((c_k')_{k\in\mathcal{J}}\right), \Psi_j(m)\right) &\equiv \widehat{b_j}\left(\Gamma_j\left(\varphi_k(c_k')_{k\in\mathcal{J}}\right), \Psi_j\left(\pi^{(c_k')_{k\in\mathcal{J}}}(m)\right)\right). \end{aligned}$$

Therefore, $(c, \widehat{s}, t^*) \in \mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^I, C^{II}\right]}$ replicates $(c, s, t) \in \mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^I, C^{IV}\right]}$.■

A.2 Proof of Proposition 1

We need the following lemmas to prove Proposition 1, and the proofs are relegated to Appendix A.2.1 and A.2.2.

Lemma 3 *Given* $\langle\mathcal{A},\ \Gamma,\ \Psi\rangle = \left\langle\mathcal{A}^{non-delegated},\ \Gamma^{private},\ \Psi^{private}\right\rangle$, *we have*

$$Z^{\mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} \subset Z^{\mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{P}, C^{\mathcal{A}}\right]}}.$$

Lemma 4 *Given* $\mathcal{A} = \mathcal{A}^{non-delegated}$, *we have*

$$Z^{\mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} \subset Z^{\mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{R}, C^{\mathcal{A}}\right]}},$$

$$\forall\langle\Gamma,\ \Psi\rangle \in \left\{\left\langle\Gamma^{public}, \Psi^{private}\right\rangle,\ \left\langle\Gamma^{public}, \Psi^{public}\right\rangle\right\}.$$

Proof of Proposition 1. Given $\mathcal{A} = \mathcal{A}^{non-delegated}$, Lemma 1 and $C^{\mathcal{A}} \sqsupset^* C^R \sqsupset^* C^P$ imply

$$Z^{\mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} \supset Z^{\mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{R}, C^{\mathcal{A}}\right]}} \supset Z^{\mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{P}, C^{\mathcal{A}}\right]}},$$

$$\forall\langle\Gamma,\ \Psi\rangle \in \left\{\left\langle\Gamma^{private}, \Psi^{private}\right\rangle,\ \left\langle\Gamma^{public}, \Psi^{private}\right\rangle,\ \left\langle\Gamma^{public}, \Psi^{public}\right\rangle\right\},$$

which, together with Lemmas 3 and 4, implies Proposition 1.■

A.2.1 Proof of Lemma 3

Fix $\langle \mathcal{A},\ \Gamma,\ \Psi\rangle = \left\langle \mathcal{A}^{non-delegated},\ \Gamma^{private},\ \Psi^{private}\right\rangle$. Fix any $(c,s,t) \in \mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}},C^{\mathcal{A}}\right]}$. We aim to replicate (c,s,t) with a $\left[C^{P},C^{\mathcal{A}}\right]$-equilibrium that induces the allocation $z^{(c,s,t)}$.

Replicating c with $c^{(c,s,t)} \in C^{P}$:

On the equilibrium path, for each $j \in \mathcal{J}$, offering c_j is equivalent to offering the menu contract, $c_j^{(c,s,t)} : M_j^{(c,s,t)} \longrightarrow 2^{Y_j} \diagdown \{\varnothing\}$ with

$$M_j^{(c,s,t)} \equiv \left\{t_j\left[\Gamma_j\left(\left(c_j,c'_{-j}\right)\right), \Psi_j\left(s\left(\left(c_j,c'_{-j}\right),\theta\right)\right)\right] : \left(c'_{-j},\theta\right) \in C_{-j}^{\mathcal{A}} \times \Theta\right\},$$
$$c_j^{(c,s,t)}\left(y_j\right) = y_j,\ \forall y_j \in M_j^{(c,s,t)}.$$

Given $\left(\left(c_j,c'_{-j}\right),\theta\right) \in C^{\mathcal{A}} \times \Theta$, if all players follow (s,t), the subset $E_j = c_j\left[s_j\left(\left(c_j,c'_{-j}\right),\theta\right)\right]$ is pinned down for j at Stage 2, and j takes the action $y_j = t_j\left[\Gamma_j\left(\left(c_j,c'_{-j}\right)\right), \Psi_j\left(s\left(\left(c_j,c'_{-j}\right),\theta\right)\right)\right]$ at Stage 3. $M_j^{(c,s,t)}$ is the set of all such $y_j = t_j\left[\Gamma_j\left(\left(c_j,c'_{-j}\right)\right), \Psi_j\left(s\left(\left(c_j,c'_{-j}\right),\theta\right)\right)\right]$.

In the $\left[C^{\mathcal{A}},C^{\mathcal{A}}\right]$-game, $C^{\mathcal{A}}$ and $M^{\mathcal{A}}$ are the contract space and the message space, respectively. Define $c^{(c,s,t)} \equiv \left(c_k^{(c,s,t)}\right)_{k\in\mathcal{J}} \in C^{P}$ and let

$$\begin{aligned}\widehat{C} &\equiv \times_{k\in\mathcal{J}}\widehat{C}_k \equiv \times_{k\in\mathcal{J}}\left(\left\{c_k^{(c,s,t)}\right\} \cup C_k^{\mathcal{A}}\right),\\ \widehat{M} &\equiv \times_{k\in\mathcal{J}}\widehat{M}_k \equiv \times_{k\in\mathcal{J}}\left(M_k^{(c,s,t)} \cup M_k^{\mathcal{A}}\right)\end{aligned}$$

denote the relevant contract space and message space in the $\left[C^{P},C^{\mathcal{A}}\right]$-game.

Replicating $s \equiv (s_k)_{k\in\mathcal{J}}$ with $s^{(c,s,t)} \equiv \left(s_k^{(c,s,t)}\right)_{k\in\mathcal{J}}$:

When the agent observes $c_k^{(c,s,t)}$ in the $\left[C^{P},C^{\mathcal{A}}\right]$-game, he interprets it as c_k in the $\left[C^{\mathcal{A}},C^{\mathcal{A}}\right]$-game, due to the replication process above. Also, when the agent observes $c'_j \in C_j^{\mathcal{A}}$ in the $\left[C^{P},C^{\mathcal{A}}\right]$-game, he interprets it as $c'_j \in C_j^{\mathcal{A}}$ in the $\left[C^{\mathcal{A}},C^{\mathcal{A}}\right]$-game. To record this interpretation, define $\gamma_j : \widehat{C}_j \longrightarrow C_j^{\mathcal{A}}$ for each $j \in \mathcal{J}$ as

$$\gamma_j\left(\widehat{c}_j\right) \equiv \begin{cases} c_j, & \text{if } \widehat{c}_j = c_j^{(c,s,t)};\\ \widehat{c}_j, & \text{if } \widehat{c}_j \in C_j^{\mathcal{A}},\end{cases}$$

and denote $\gamma\left(\widehat{c}\right) \equiv \left(\left[\gamma_k\left(\widehat{c}_k\right)\right]_{k\in\mathcal{J}}\right) \in C^{\mathcal{A}}$ for all $\widehat{c} = \left(\widehat{c}_k\right)_{k\in\mathcal{J}} \in \widehat{C}$.

For the agent's strategy in the $\left[C^{P},C^{\mathcal{A}}\right]$-game, we replicate $s \equiv (s_k)_{k\in\mathcal{J}}$ with $s^{(c,s,t)} \equiv \left(s_k^{(c,s,t)}\right)$ defined as follows. For each $j \in \mathcal{J}$ and each $\left(\widehat{c},\theta\right) \in \widehat{C} \times \Theta$,

$$s_j^{(c,s,t)}\left(\widehat{c},\theta\right) = \begin{cases} s_j\left[\gamma\left(\widehat{c}\right),\ \theta\right], & \text{if } \widehat{c}_j \in C_j^{\mathcal{A}};\\ t_j\left(\Gamma_j\left(\gamma\left(\widehat{c}\right)\right), \Psi_j\left[s\left(\gamma\left(\widehat{c}\right),\theta\right)\right]\right), & \text{if } \widehat{c}_j = c_j^{(c,s,t)}.\end{cases}$$

When principals offer $\widehat{c}$ at Stage 1 in the $\left[C^P, C^{\mathcal{A}}\right]$-game, the agent regards it as $\gamma(\widehat{c})$ in the $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-game. Given $\gamma(\widehat{c})$ in the $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-game, the agent sends $s_j[\gamma(\widehat{c}),\ \theta]$ to j at Stage 2, and at Stage 3, j would choose the action $t_j(\Gamma_j(\gamma(\widehat{c})), \Psi_j[s(\gamma(\widehat{c}), \theta)])$. Then, in the $\left[C^P, C^{\mathcal{A}}\right]$-game, $s_j^{(c,s,t)}$ replicates s_j: if $\widehat{c}_j \in C_j^{\mathcal{A}}$, the agent sends $s_j[\gamma(\widehat{c}),\ \theta]$ to j; if $\widehat{c}_j = c_j^{(c,s,t)} \in C_j^P$, the agent chooses the action

$$t_j(\Gamma_j(\gamma(\widehat{c})), \Psi_j[s(\gamma(\widehat{c}), \theta)]) \in c_j^{(c,s,t)}.$$

<u>Replicating (b_j, t_j) with $\left(b_j^{(c,s,t)}, t_j^{(c,s,t)}\right)$:</u>

For each $j \in \mathcal{J}$, consider

$$Q_j \equiv \left\{[\Gamma_j(\widehat{c}), \Psi_j(\widehat{m})] : \widehat{c} \in \widehat{C} \text{ and } \widehat{m} \in \widehat{M}\right\},$$

$$Q_j^* \equiv \left\{[\Gamma_j(\widehat{c}), \Psi_j(\widehat{m})] \in Q_j : \widehat{c}_j = c_j^{(c,s,t)}\right\},$$

$$Q_j^{**} \equiv \left\{\left[\Gamma_j(\widehat{c}), \Psi_j\left(s^{(c,s,t)}[\widehat{c}, \theta]\right)\right] \in Q_j : \widehat{c}_j \in C_j^{\mathcal{A}},\ \widehat{c}_{-j} = c_{-j}^{(c,s,t)} \text{ and } \theta \in \Theta\right\},$$

i.e., in the $\left[C^P, C^{\mathcal{A}}\right]$-game, Q_j is the set of all possible information that principal j may observe before j takes an action at Stage 3; Q_j^* is the subset of information with which principal j offers the menu contract $c_j^{(c,s,t)}$; Q_j^{**} is the subset of information with which principal j offers $\widehat{c}_j \in C_j^{\mathcal{A}}$ and cannot confirm that the other players have deviated.

First, when principal j observes $[\Gamma_j(\widehat{c}), \Psi_j(\widehat{m})] \in Q_j^*$, we have $\widehat{c}_j = c_j^{(c,s,t)}$, which is a menu contract (i.e., a delegated contract). As a result, j's decision at Stage 3 is degenerate and the belief is irrelevant. Second, when principal j observes $[\Gamma_j(\widehat{c}), \Psi_j(\widehat{m})] \in Q_j \diagdown \left(Q_j^* \cup Q_j^{**}\right)$, principal j is on an off-equilibrium path and $c_j' \in C_j^{\mathcal{A}}$. Define

$$\begin{aligned} t_j^{(c,s,t)}[\Gamma_j(\widehat{c}), \Psi_j(\widehat{m})] &\equiv t_j[\Gamma_j(\widehat{c}), \Psi_j(\widehat{m})], \\ b_j^{(c,s,t)}[\Gamma_j(\widehat{c}), \Psi_j(\widehat{m})] &\equiv b_j[\Gamma_j(\widehat{c}), \Psi_j(\widehat{m})], \end{aligned}$$

i.e., $\left(b_j^{(c,s,t)}, t_j^{(c,s,t)}\right)$ copies (b_j, t_j).

Finally, when principal j observes $q_j = \left[\Gamma_j\left(\widehat{c}_j, c_{-j}^{(c,s,t)}\right), \Psi_j\left(s^{(c,s,t)}\left[\left(\widehat{c}_j, c_{-j}^{(c,s,t)}\right), \theta\right]\right)\right] \in Q_j^{**}$, i.e., j cannot confirm that the other players have deviated. Define

$$t_j^{(c,s,t)}(q_j) \equiv t_j[\Gamma_j(\widehat{c}_j, c_{-j}), \Psi_j(s[(\widehat{c}_j, c_{-j}), \theta])],$$

$$b_j^{(c,s,t)}(q_j)\left[\left\{\left(\widehat{c}_j, c_{-j}^{(c,s,t)}\right)\right\} \times \widehat{M} \times \Theta\right] = 1 \text{ and}$$

$$b_j^{(c,s,t)}(q_j)\left[\left\{\left[\left(\widehat{c}_j, c_{-j}^{(c,s,t)}\right),\ s^{(c,s,t)}\left[\left(\widehat{c}_j, c_{-j}^{(c,s,t)}\right), \theta'\right],\ \theta'\right]\right\}\right] \equiv \begin{cases} \frac{p(\theta')}{p[\Upsilon(\theta)]} & \text{if } \theta' \in \Upsilon(\theta); \\ 0 & \text{otherwise} \end{cases},$$

where $\Upsilon(\theta) \equiv \left\{ \widetilde{\theta} \in \Theta : \Psi_j \left(s^{(c,s,t)} \left[\left(\widehat{c}_j, c_{-j}^{(c,s,t)} \right), \widetilde{\theta} \right] \right) = \Psi_j \left(s^{(c,s,t)} \left[\left(\widehat{c}_j, c_{-j}^{(c,s,t)} \right), \theta \right] \right) \right\}$,

i.e., $b_j^{(c,s,t)}(q_j)$ believes in $\left(\widehat{c}_j, c_{-j}^{(c,s,t)}\right)$ with probability 1 and its belief on $\widehat{M} \times \Theta$ is derived by Bayes' rule.

It is straightforward to see that (c, s, t) is replicated by $\left(c^{(c,s,t)}, s^{(c,s,t)}, t^{(c,s,t)}\right)$, and all the players inherit incentive compatibility from (c, s, t), i.e., $\left(c^{(c,s,t)}, s^{(c,s,t)}, t^{(c,s,t)}\right)$ is a $\left[C^P, C^{\mathcal{A}}\right]$-equilibrium and $z^{(c,s,t)} = z^{\left(c^{(c,s,t)}, s^{(c,s,t)}, t^{(c,s,t)}\right)}$.■

A.2.2 Proof of Lemma 4

Fix $\mathcal{A} = \mathcal{A}^{non-delegated}$ and any $\langle \Gamma, \Psi \rangle \in \left\{ \left\langle \Gamma^{public}, \Psi^{private} \right\rangle, \left\langle \Gamma^{public}, \Psi^{public} \right\rangle \right\}$. Fix any $(c, s, t) \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle \text{-} \left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}$. We aim to replicate (c, s, t) with a $\left[C^R, C^{\mathcal{A}}\right]$-equilibrium that induces $z^{(c,s,t)}$.

Replicating c with $c^{(c,s,t)} \in C^R$:

On the equilibrium path, for each $j \in \mathcal{J}$, offering c_j is equivalent to offering the menu-of-menu-with-recommendation contract, $c_j^{(c,s,t)} : M_j^{(c,s,t)} \longrightarrow Y_j$ with

$$M_j^{(c,s,t)} \equiv \left\{ \left[E_j = c_j \left[s_j \left(c', \theta \right) \right], \ y_j = t_j \left[\Gamma_j \left(c' \right), \Psi_j \left(s \left(c', \theta \right) \right) \right] \right] : \begin{array}{l} c'_j = c_j, \\ \left(c'_{-j}, \theta \right) \in C_{-j}^{\mathcal{A}} \times \Theta \end{array} \right\}, \tag{26}$$

$$c_j^{(c,s,t)} \left[E_j, \ y_j \right] = E_j, \ \forall \left[E_j, \ y_j \right] \in M_j^{(c,s,t)}.$$

Given $\left(\left(c_j, c'_{-j} \right), \theta \right) \in C^{\mathcal{A}} \times \Theta$, if all players follow (s, t), the subset $E_j = c_j \left[s_j \left(\left(c_j, c'_{-j} \right), \theta \right) \right]$ is fixed for j at Stage 2, and j takes the action $y_j = t_j \left[\Gamma_j \left(\left(c_j, c'_{-j} \right) \right), \Psi_j \left(s \left(\left(c_j, c'_{-j} \right), \theta \right) \right) \right]$ at Stage 3. $M_j^{(c,s,t)}$ is the set of all such profiles. Define $c^{(c,s,t)} \equiv \left(c_k^{(c,s,t)} \right)_{k \in \mathcal{J}} \in C^R$, and let

$$\begin{aligned} \widehat{C} &\equiv \times_{k \in \mathcal{J}} \widehat{C}_k \equiv \times_{k \in \mathcal{J}} \left(\left\{ c_k^{(c,s,t)} \right\} \cup C_k^{\mathcal{A}} \right) \text{ and} \\ \widehat{M} &\equiv \times_{k \in \mathcal{J}} \widehat{M}_k \equiv \times_{k \in \mathcal{J}} \left(M_k^{(c,s,t)} \cup M_k^{\mathcal{A}} \right) \end{aligned}$$

denote the relevant contract space and the relevant message space respectively in the $\left[C^R, C^{\mathcal{A}}\right]$-game. In the $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-game, $C^{\mathcal{A}}$ and $M^{\mathcal{A}}$ are the relevant contract space and message space, respectively.

Replicating $s \equiv (s_k)_{k \in \mathcal{J}}$ with $s^{(c,s,t)} \equiv \left(s_k^{(c,s,t)} \right)_{k \in \mathcal{J}}$:

When the agent observes $c_k^{(c,s,t)}$ in the $[C^R, C^{\mathcal{A}}]$-game, he interprets it as c_k in the $[C^{\mathcal{A}}, C^{\mathcal{A}}]$-game, due to the replication process above. Also, when the agent observes $c'_j \in C_j^{\mathcal{A}}$ in the $[C^R, C^{\mathcal{A}}]$-game, he interprets it as $c'_j \in C_j^{\mathcal{A}}$ in the $[C^{\mathcal{A}}, C^{\mathcal{A}}]$-game. To record this interpretation, define $\gamma_j : \widehat{C}_j \longrightarrow C_j^{\mathcal{A}}$ for each $j \in \mathcal{J}$ as

$$\gamma_j(\widehat{c}_j) \equiv \begin{cases} c_j, & \text{if } \widehat{c}_j = c_j^{(c,s,t)}; \\ \widehat{c}_j, & \text{if } \widehat{c}_j \in C_j^{\mathcal{A}}, \end{cases} \tag{27}$$

and denote $\gamma(\widehat{c}) \equiv \left([\gamma_k(\widehat{c}_k)]_{k\in\mathcal{J}}\right) \in C^{\mathcal{A}}$ for all $\widehat{c} = (\widehat{c}_k)_{k\in\mathcal{J}} \in \widehat{C}$.

For the agent's strategy in the $[C^R, C^{\mathcal{A}}]$-game, we replicate $s \equiv (s_k)_{k\in\mathcal{J}}$ with $s^{(c,s,t)} \equiv \left(s_k^{(c,s,t)}\right)$ defined as follows. For each $j \in \mathcal{J}$ and all $(\widehat{c}, \theta) \in \widehat{C} \times \Theta$,

$$s_j^{(c,s,t)}(\widehat{c}, \theta) = \begin{cases} s_j[\gamma(\widehat{c}),\ \theta], & \text{if } \widehat{c}_j \in C_j^{\mathcal{A}}; \\ [E_j = c_j(s_j[\gamma(\widehat{c}),\ \theta]),\ y_j = t_j(\Gamma_j(\gamma(\widehat{c})), \Psi_j[s(\gamma(\widehat{c}), \theta)])], & \text{if } \widehat{c}_j = c_j^{(c,s,t)}. \end{cases}$$

I.e., when principals offer $\widehat{c}$ in the $[C^R, C^{\mathcal{A}}]$-game, the agent regards it as $\gamma(\widehat{c})$ in the $[C^{\mathcal{A}}, C^{\mathcal{A}}]$-game. Given $\gamma(\widehat{c})$ in the $[C^{\mathcal{A}}, C^{\mathcal{A}}]$-game, the agent sends $s_j[\gamma(\widehat{c}),\ \theta]$ to j at Stage 2, which pins down the subset $c_j(s_j[\gamma(\widehat{c}),\ \theta])$ for j, and j takes the action $t_j(\Gamma_j(\gamma(\widehat{c})), \Psi_j[s(\gamma(\widehat{c}), \theta)])$ at Stage 3. Then, in the $[C^R, C^{\mathcal{A}}]$-game, $s_j^{(c,s,t)}$ replicates s_j: if $\widehat{c}_j \in C_j^{\mathcal{A}}$, the agent sends $s_j[\gamma(\widehat{c}),\ \theta]$ to j; if $\widehat{c}_j = c_j^{(c,s,t)} \in C_j^R$, the agent chooses the subset $c_j(s_j[\gamma(\widehat{c}),\ \theta])$ with the recommendation $t_j(\Gamma_j(\gamma(\widehat{c})), \Psi_j[s(\gamma(\widehat{c}), \theta)])$.

<u>Replicating (b_j, t_j) with $\left(b_j^{(c,s,t)}, t_j^{(c,s,t)}\right)$:</u>

We aim to replicate an equilibrium in the $[C^{\mathcal{A}}, C^{\mathcal{A}}]$-game with an equilibrium in the $[C^R, C^{\mathcal{A}}]$-game. $\gamma \equiv (\gamma_k)_{k\in\mathcal{J}}$ defined in (27) describes how players translate contracts in the the $[C^R, C^{\mathcal{A}}]$-game to contracts in the $[C^{\mathcal{A}}, C^{\mathcal{A}}]$-game. We still need to define how players translate messages in the the $[C^R, C^{\mathcal{A}}]$-game to messages in the $[C^{\mathcal{A}}, C^{\mathcal{A}}]$-game.

By (26), there exists a surjective function $\zeta_j : C_{-j}^{\mathcal{A}} \times \Theta \longrightarrow M_j^{(c,s,t)}$:

$$\zeta_j(c'_{-j}, \theta) = \left[E_j = c_j\left[s_j\left((c_j, c'_{-j}), \theta\right)\right],\ y_j = t_j\left[\Gamma_j\left((c_j, c'_{-j})\right), \Psi_j\left(s\left((c_j, c'_{-j}), \theta\right)\right)\right]\right],$$

i.e., upon observing $\left((c_j, c'_{-j}), \theta\right)$, by following s, the agent's message would pin down the subset $E_j = c_j\left[s\left((c_j, c'_{-j}), \theta\right)\right] \subset Y_j$ at Stage 2, and by following t_j, principal j would take the action $y_j = t_j\left[\Gamma_j\left((c_j, c'_{-j})\right), \Psi_j\left(s\left((c_j, c'_{-j}), \theta\right)\right)\right]$ at Stage 3.—$\zeta(c'_{-j}, \theta)$ records this profile, $[E_j,\ y_j]$.

Fix any injective $\zeta_j^{-1} : M_j^{(c,s,t)} \longrightarrow C_{-j}^{\mathcal{A}} \times \Theta$ such that

$$\zeta_j\left[\zeta_j^{-1}(m_j)\right] = m_j, \forall m_j \in M_j^{(c,s,t)},$$

i.e., each $m_j \in M_j^{(c,s,t)}$ is mapped to some $\left(c'_{-j}, \theta\right)$ such that $\zeta_j\left(c'_{-j}, \theta\right) = m_j$. That is, upon observing $m_j \in M_j^{(c,s,t)}$ in the $\left[C^R, C^{\mathcal{A}}\right]$-game, principals interpret it as the message $s_j\left[c_j, \zeta_j^{-1}(m_j)\right]$ in the $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-game.

Given public announcement, the contract profile offered at Stage 1 will be common knowledge. Given $\widehat{c} \in \widehat{C}$ offered at Stage 1, consider

$$M_j^{\widehat{c}} \equiv \left\{ m_j \in \widehat{M}_j : \begin{array}{l} \widehat{c}_j \in C_j^{\mathcal{A}} \Longrightarrow m_j \in M_j^{\mathcal{A}}, \\ \widehat{c}_j = c_j^{(c,s,t)} \Longrightarrow m_j \in M_j^{(c,s,t)} \end{array} \right\} \text{ and } M^{\widehat{c}} \equiv \left(M_j^{\widehat{c}}\right)_{j \in \mathcal{J}}.$$

i.e., $M^{\widehat{c}}$ is the set of message profiles that could be sent by the agent at Stage 2 in the $\left[C^R, C^{\mathcal{A}}\right]$-game. Given $\widehat{c} \in \widehat{C}$ offered at Stage 1, the function $\tau^{\widehat{c}} \equiv \left(\tau_j^{\widehat{c}} : M_j^{\widehat{c}} \longrightarrow M_j^{\mathcal{A}}\right)_{j \in \mathcal{J}}$ describes how the players translate messages in the $\left[C^R, C^{\mathcal{A}}\right]$-game to the messages in the $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-game.

$$\tau_j^{\widehat{c}}(m_j) \equiv \begin{cases} m_j, & \text{if } \widehat{c}_j \in C_j^{\mathcal{A}}; \\ s_j\left(c_j, \zeta^{-1}(m_j)\right), & \text{if } \widehat{c}_j = c_j^{(c,s,t)}, \end{cases}, \forall j \in \mathcal{J},$$

i.e., the players re-label m_j in the $\left[C^R, C^{\mathcal{A}}\right]$-game if and only if $\widehat{c}_j = c_j^{(c,s,t)}$, and when $\widehat{c}_j = c_j^{(c,s,t)}$, a message m_j in the $\left[C^R, C^{\mathcal{A}}\right]$-game is interpreted as $s_j\left(c_j, \zeta_j^{-1}(m_j)\right)$ in the $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-game.

We are now ready to replicate t_j with $t_j^{(c,s,t)}$.

$$t_j^{(c,s,t)}\left[\Gamma_j(\widehat{c}), \Psi_j(\widehat{m})\right] \equiv t_j\left[\Gamma_j(\gamma(\widehat{c})), \Psi_j\left(\tau^{\widehat{c}}(\widehat{m})\right)\right], \forall (\widehat{c}, \widehat{m}) \in \widehat{C} \times \widehat{M},$$

i.e., players translate the profile $(\widehat{c}, \widehat{m}) \in \widehat{C} \times \widehat{M}$ in the $\left[C^R, C^{\mathcal{A}}\right]$-game to the profile $\left(\gamma(\widehat{c}), \tau^{\widehat{c}}(\widehat{m})\right) \in C^{\mathcal{A}} \times M^{\mathcal{A}}$, and $t_j^{(c,s,t)}$ replicates t_j.

Similarly, we replicate b_j with $b_j^{(c,s,t)}$, subject to the translation of $\left(\gamma, \tau^{\widehat{c}}\right)$. Rigorously, consider

$$Q_j \equiv \left\{\left[\Gamma_j(\widehat{c}), \Psi_j(\widehat{m})\right] : \widehat{c} \in \widehat{C} \text{ and } \widehat{m} \in \widehat{M}\right\},$$

$$Q_j^* \equiv \left\{\left[\Gamma_j(\widehat{c}), \Psi_j(\widehat{m})\right] \in Q_j : \widehat{c}_{-j} = c_{-j}^{(c,s,t)} \text{ and } \widehat{m} = s^{(c,s,t)}\left[\widehat{c}, \theta\right] \text{ for some } \theta \in \Theta\right\}.$$

i.e., Q_j is the set of all possible information that principal j may observe before j takes an action at Stage 3; Q_j^* is the subset of information by which j cannot confirm that the other players have deviated.

When principal j observes $q_j = \left[\Gamma_j\left(\widehat{c}_j, c_{-j}^{(c,s,t)}\right), \Psi_j\left(s^{(c,s,t)}\left[\left(\widehat{c}_j, c_{-j}^{(c,s,t)}\right), \theta\right]\right)\right] \in Q_j^*$ for some $\theta \in \Theta$, j's belief is induced by Bayes' rule, i.e.,

$$b_j^{(c,s,t)}(q_j)\left[\left\{\left[\left(\widehat{c}_j, c_{-j}^{(c,s,t)}\right),\ s^{(c,s,t)}\left[\left(\widehat{c}_j, c_{-j}^{(c,s,t)}\right), \theta'\right],\ \theta'\right]\right\}\right] \equiv \begin{cases} \frac{p(\theta')}{p[\Upsilon(\theta)]} & \text{if } \theta' \in \Upsilon(\theta); \\ \\ 0 & \text{otherwise} \end{cases},$$

where

$$\Upsilon(\theta) \equiv \left\{\widetilde{\theta} \in \Theta : \Psi_j\left(s^{(c,s,t)}\left[\left(\widehat{c}_j, c_{-j}^{(c,s,t)}\right), \widetilde{\theta}\right]\right) = \Psi_j\left(s^{(c,s,t)}\left[\left(\widehat{c}_j, c_{-j}^{(c,s,t)}\right), \theta\right]\right)\right\}.$$

When principal j observes $q_j = [\Gamma_j(\widehat{c}), \Psi_j(\widehat{m})] \in Q_j \diagdown Q_j^*$, define

$$b_j^{(c,s,t)}(q_j)\left[\{\widehat{c}\} \times \widehat{M} \times \Theta\right] = 1,$$

$$\begin{aligned} & b_j^{(c,s,t)}[\Gamma_j(\widehat{c}), \Psi_j(\widehat{m})][\{\widehat{c}\} \times E \times \{\theta'\}] \\ \equiv\ & b_j\left[\Gamma_j(\gamma(\widehat{c})), \Psi_j\left(\tau^{\widehat{c}}(\widehat{m})\right)\right]\left[\{\gamma(\widehat{c})\} \times \tau^{\widehat{c}}(E) \times \{\theta'\}\right], \forall E \in 2^{\widehat{M}}. \end{aligned}$$

It is straightforward to see that (c, s, t) is replicated by $\left(c^{(c,s,t)}, s^{(c,s,t)}, t^{(c,s,t)}\right)$, and all the players inherit incentive compatibility from (c, s, t), i.e., $\left(c^{(c,s,t)}, s^{(c,s,t)}, t^{(c,s,t)}\right)$ is a $\left[C^P, C^{\mathcal{A}}\right]$-equilibrium and $z^{(c,s,t)} = z^{\left(c^{(c,s,t)}, s^{(c,s,t)}, t^{(c,s,t)}\right)}$.■

A.3 Proof of Proposition 2

Fix $\mathcal{A} = \mathcal{A}^{non-delegated}$. Fix any $\langle\Gamma,\ \Psi\rangle \in \left\{\left\langle\Gamma^{private}, \Psi^{private}\right\rangle,\ \left\langle\Gamma^{public}, \Psi^{private}\right\rangle,\ \left\langle\Gamma^{public}, \Psi^{public}\right\rangle\right\}$ and any $I \in \{\mathcal{A}, P, F, R\}$. Since $C^{\mathcal{A}} \sqsupset^{**} C^F$, Lemma 2 implies $Z^{\mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^I, C^{\mathcal{A}}\right]}} \supset Z^{\mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^I, C^F\right]}}$. We now prove

$$Z^{\mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^I, C^{\mathcal{A}}\right]}} \subset Z^{\mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^I, C^F\right]}}.$$

Fix any $(c, s, t) \in \mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^I, C^{\mathcal{A}}\right]}$. We will replicate (c, s, t) with some $\left(c, \overline{s}, \overline{t}\right) \in \mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^I, C^F\right]}$ such that $z^{(c,s,t)} = z^{\left(c, \overline{s}, \overline{t}\right)}$.

Replicating $s \equiv (s_k)_{k\in\mathcal{J}}$ with $\overline{s} \equiv (\overline{s}_k)_{k\in\mathcal{J}}$:

We replicate s in the $\left[C^I, C^{\mathcal{A}}\right]$-game with $\overline{s}$ in the $\left[C^I, C^F\right]$-game. Since s is defined on $\left(\{c_k\} \cup C_k^{\mathcal{A}}\right)_{k\in\mathcal{J}}$ and $\overline{s}$ is defined on $\left(\{c_k\} \cup C_k^F\right)_{k\in\mathcal{J}}$, we need to define a function γ : $\left(\{c_k\} \cup C_k^F\right)_{k\in\mathcal{J}} \longrightarrow \left(\{c_k\} \cup C_k^{\mathcal{A}}\right)_{k\in\mathcal{J}}$, such that, upon observing $c' \in \left(\{c_k\} \cup C_k^F\right)_{k\in\mathcal{J}}$ in the $\left[C^I, C^F\right]$-game, the agent regards it as $\gamma(c')$ in the $\left[C^I, C^{\mathcal{A}}\right]$-game, and then, $\overline{s}(c')$ mimics $s[\gamma(c')]$. Thus, for each $j \in \mathcal{J}$, consider $\gamma_j : \{c_j\} \cup C_j^F \longrightarrow \{c_j\} \cup C_j^{\mathcal{A}}$ such that

$$\gamma_j(c_j) = c_j,$$

and for each $c_j' \in C_j^F \diagdown \{c_j\}$, we define $\gamma_j\left(c_j'\right)$ as follows. Since $c_j' \in C_j^F$, we have a pair of $\left[L_j \subset 2^{Y_j} \text{ and } H_j = \{[E_j, y_j] : y_j \in E_j\}\right]$ satisfying Definition 4. Fix any injective function $\phi_j^{c_j'} : L_j \cup H_j \longrightarrow M_j^{\mathcal{A}}$, i.e., given $c_j' \in C_j^F$, we identify a message $m_j' \in L_j \cup H_j$ in the $\left[C^I, C^F\right]$-game to the message $\phi_j^{c_j'}\left(m_j'\right) \in M_j^{\mathcal{A}}$ in the $\left[C^I, C^{\mathcal{A}}\right]$-game. With slight abuse of notation, let $\left(\phi_j^{c_j'}\right)^{-1} : \phi_j^{c_j'}\left[L_j \cup H_j\right] \longrightarrow L_j \cup H_j$ denote the inverse function of $\phi_j^{c_j'}$, i.e.,

$$\phi_j^{c_j'}\left[\left(\phi_j^{c_j'}\right)^{-1}(m_j)\right] = m_j, \forall m_j \in \phi_j^{c_j'}\left[L_j \cup H_j\right].$$

We thus define $\gamma_j\left(c_j'\right)$ for each $c_j' \in C_j^F \diagdown \{c_j\}$ as follows.

$$\gamma_j\left(c_j'\right)[m_j] = \begin{cases} c_j'\left(\left(\phi_j^{c_j'}\right)^{-1}[m_j]\right) & \text{if } m_j \in \phi_j^{c_j'}\left[L_j \cup H_j\right]; \\ E_j & \text{otherwise,} \end{cases} \quad \forall m_j \in M_j^{\mathcal{A}},$$

i.e., we first embed the message space $L_j \cup H_j$ into $M_j^{\mathcal{A}}$ by $\phi_j^{c_j'}$; second, we copy c_j' with $\gamma_j\left(c_j'\right)$ on the embedded message space; third, all of the other messages in $M_j^{\mathcal{A}}$ are mapped to E_j. Furthermore, denote $\gamma(c') \equiv (\gamma_k(c_k'))_{k\in\mathcal{J}}$ and $\phi^{c'} :\equiv \left[\phi_j^{c_j'} : L_j \cup H_j \longrightarrow M_j^{\mathcal{A}}\right]_{k\in\mathcal{J}}$.

We are now ready to define $\overline{s} \equiv (\overline{s}_k)_{k\in\mathcal{J}}$. For each $j \in \mathcal{J}$, define

$$\overline{s}_j(c', \theta) \equiv \begin{cases} s_j(\gamma(c'), \theta) & \text{if } c_j' = c_j; \\ \left(\phi_j^{c_j'}\right)^{-1}\left[s_j(\gamma(c'), \theta)\right] & \text{if } c_j' \neq c_j \text{ and } s_j(\gamma(c'), \theta) \in \phi_j^{c_j'}[L_j]; \\ \left[E_j,\ y_j = t_j\left(\Gamma_j(\gamma(c')), \Psi_j(s(\gamma(c'), \theta))\right)\right] & \text{otherwise.} \end{cases}$$

When the agent observes $c' \in \left(\{c_k\} \cup C_k^F\right)_{k\in\mathcal{J}}$ in the $\left[C^I, C^F\right]$-game, the agent translates it to $\gamma(c') \in \left(\{c_k\} \cup C_k^{\mathcal{A}}\right)_{k\in\mathcal{J}}$ being offered in the $\left[C^I, C^{\mathcal{A}}\right]$-game. Then, $\overline{s}_j(c', \theta)$ replicates $s_j(\gamma(c'), \theta)$: if $c_j' = c_j$, we have $\overline{s}_j(c', \theta) = s_j(\gamma(c'), \theta)$; if $c_j' \neq c_j$ and $s_j(\gamma(c'), \theta) \in \phi_j^{c_j'}[L_j]$, $\overline{s}_j(c', \theta)$ mimics $s_j(\gamma(c'), \theta)$ subject to re-labeling of message by $\left(\phi_j^{c_j'}\right)^{-1}$; otherwise, the message $s_j(\gamma(c'), \theta)$ pins down the subset E_j at Stage 2 in the $\left[C^I, C^{\mathcal{A}}\right]$-game, and at stage 3, j would take the action $t_j\left(\Gamma_j(\gamma(c')), \Psi_j(s(\gamma(c'), \theta))\right)$, and hence, $\overline{s}_j(c', \theta)$ mimics this by choosing

$$\left[E_j,\ y_j = t_j\left(\Gamma_j(\gamma(c')), \Psi_j(s(\gamma(c'), \theta))\right)\right]$$

in the menu-of-menu-with-full-recommendation contract $c_j' \in C_j^F$.

Replicating (b_j, t_j) with $\left(\overline{b}_j, \overline{t}_j\right)$:

Given $c_j' \in \{c_j\} \cup C_j^F$, let $M_j^{c_j'}$ denote the domain of c_j'. For each $j \in \mathcal{J}$, define

$$Q_j^F \equiv \left\{ (\Gamma_j(c'), \Psi_j(m)) : \begin{array}{c} c' = (c_k')_{k\in\mathcal{J}} \in \times_{k\in\mathcal{J}} \left(\{c_k\} \cup C_k^F\right) \\ m = (m_k)_{k\in\mathcal{J}} \in \times_{k\in\mathcal{J}} \left(M_k^{c_k'}\right) \end{array} \right\},$$

$$Q_j^{F*} \equiv \left\{ (\Gamma_j(c'), \Psi_j(\overline{s}(c', \theta))) : \begin{array}{c} c' \in \times_{k\in\mathcal{J}} \left(\{c_k\} \cup C_k^F\right) \\ \theta \in \Theta \end{array} \right\},$$

i.e., in the $[C^I, C^F]$-game, Q_j^F is the set of all possible information that principal j may observe before j takes an action at Stage 3, and Q_j^{F*} is the subset of information induced by the agent following $\overline{s}$ (which replicates s). For each $q_j \in Q_j^{F*}$, fix any (c^{q_j}, θ^{q_j}) such that

$$q_j = (\Gamma_j(c^{q_j}), \Psi_j(\overline{s}(c^{q_j}, \theta^{q_j}))).$$

For each $q_j \in Q_j^F \diagdown Q_j^{F*}$, fix any $(\overline{c^{q_j}}, \overline{m})$ such that

$$q_j = (\Gamma_j(\overline{c^{q_j}}), \Psi_j(\overline{m})).$$

We record this as $\Sigma_j : Q_j^F \longrightarrow Q_j^{\mathcal{A}}$ such that

$$\Sigma_j(q_j) \equiv \begin{cases} [\Gamma_j(\gamma(c^{q_j})), \Psi_j(s(\gamma(c^{q_j}), \theta^{q_j}))] & \text{if } q_j \in Q_j^{F*}; \\ \\ \left[\Gamma_j[\gamma(\overline{c^{q_j}})], \Psi_j\left[\phi^{\overline{c^{q_j}}}(\overline{m})\right]\right] & \text{if } q_j \in Q_j^F \diagdown Q_j^{F*} \end{cases},$$

$$\text{where } Q_j^{\mathcal{A}} \equiv \left\{ (\Gamma_j(c'), \Psi_j(m)) : \begin{array}{c} c' = (c_k')_{k\in\mathcal{J}} \in \times_{k\in\mathcal{J}} \left(\{c_k\} \cup C_k^{\mathcal{A}}\right) \\ m = (m_k)_{k\in\mathcal{J}} \in \times_{k\in\mathcal{J}} \left(M_k^{c_k'}\right) \end{array} \right\},$$

i.e., principal j's information at Stage 3, $q_j \in Q_j^F$, in the $[C^I, C^F]$-game is translated to $\Sigma_j(q_j) \in Q_j^{\mathcal{A}}$, in the $[C^I, C^A]$-game. Thus, we define

$$\overline{t}_j(q_j) \equiv t_j(\Sigma_j(q_j)).$$

Furthermore, $\overline{b}_j(q_j)$ replicates $b_j(\Sigma_j(q_j))$, subject to re-labeling of contracts and messages. Rigorously, we define the beliefs for $\langle \Gamma^{private}, \Psi^{private} \rangle$, $\langle \Gamma^{public}, \Psi^{private} \rangle$ and $\langle \Gamma^{public}, \Psi^{public} \rangle$ as follows.

Case 1: $\langle \Gamma^{public}, \Psi^{public} \rangle$ Given $\langle \Gamma^{public}, \Psi^{public} \rangle$, all principals observe all contracts and all messages. Thus, each principal has a degenerate belief on $C \times M$, and hence, we only describe the marginal belief on Θ. Define

$$\overline{b}_j(q_j)[\{(\theta)\}] \equiv b_j(\Sigma_j(q_j))[\{(\theta)\}],$$

$\overline{b}_j(q_j)$ mimics $b_j(\Sigma_j(q_j))$ on the belief on Θ.

Case 2: $\langle \Gamma^{private}, \Psi^{private} \rangle$ For each $j \in \mathcal{J}$ and each $y_j \in Y_j$, let $c_j^{y_j}$ denote the degenerate menu contract $\{y_j\}$, i.e., it is a menu containing only y_j. Let $m_j^{y_j} = y_j$ be the unique message in this contract. Clearly, $c_j^{y_j} \in C_j^P \subset C_j^F$.

Given $\langle \Gamma^{private}, \Psi^{private} \rangle$, each principal j observes only $\left(c'_j, m'_j\right)$, and for notational simplicity, we write $t_j\left[c'_j, m'_j\right]$ for $t_j\left(\Gamma_j\left(c'\right), \Psi_j\left(m'\right)\right)$. Furthermore, j has a degenerate belief on $C_j \times M_j$, and hence, we describe only the marginal belief on $C_{-j} \times M_{-j} \times \Theta$. For each $q_j \in Q_j^F$, define

$$\begin{aligned}
&\bar{b}_j\left(q_j\right)\left[\left\{\left(\left(c_k^{y_k}\right)_{k \in \mathcal{J} \setminus \{j\}}, \left(m_k^{y_k}\right)_{k \in \mathcal{J} \setminus \{j\}}, \theta\right)\right\}\right] \\
&\equiv b_j\left(\Sigma_j\left(q_j\right)\right)\left[\left\{\left(\left(c'_k\right)_{k \in \mathcal{J} \setminus \{j\}}, \left(m'_k\right)_{k \in \mathcal{J} \setminus \{j\}}, \theta\right) : \left(y_k\right)_{k \in \mathcal{J} \setminus \{j\}} = \left(t_k\left[c'_k, m'_k\right]\right)_{k \in \mathcal{J} \setminus \{j\}}\right\}\right], \\
&\forall \left[\left(y_k\right)_{k \in \mathcal{J} \setminus \{j\}}, \theta\right] \in \left[\times_{k \in \mathcal{J} \setminus \{j\}} Y_k\right] \times \Theta,
\end{aligned}$$

i.e., $\bar{b}_j\left(q_j\right)$ mimics $b_j\left(\Sigma_j\left(q_j\right)\right)$ regarding induced belief on $\left[\times_{k \in \mathcal{J} \setminus \{j\}} Y_k\right] \times \Theta$.

Case 3: $\left\langle \Gamma^{public}, \Psi^{private} \right\rangle$ Given $\left\langle \Gamma^{public}, \Psi^{private} \right\rangle$, each principal j observes only $\left(c', m'_j\right)$, and for notational simplicity, we write $t_j\left[c', m'_j\right]$ for $t_j\left(\Gamma_j\left(c'\right), \Psi_j\left(m'\right)\right)$. Furthermore, j has a degenerate belief on $C \times M_j$, and hence, we describe only the marginal belief on $M_{-j} \times \Theta$. For each $q_j = \left(\Gamma_j\left(c^{q_j}\right), \Psi_j\left(\bar{s}\left(c^{q_j}, \theta^{q_j}\right)\right)\right) \in Q_j^{F*}$, $\bar{b}_j\left(q_j\right)$ mimics

$$b_j\left(\Sigma_j\left(q_j\right)\right) = b_j\left(\left[\Gamma_j\left(\gamma\left(c^{q_j}\right)\right), \Psi_j\left(s\left(\gamma\left(c^{q_j}\right), \theta^{q_j}\right)\right)\right]\right).$$

For each $c' \in \left(\{c_k\} \cup C_k^F\right)_{k \in \mathcal{J}}$ and each $j \in \mathcal{J}$, we use $\eta_j^{c'} : M_j^{\gamma_j\left(c'_j\right)} \longrightarrow M_j^{c'_j}$ defined below to translate messages in $M_j^{\gamma_j\left(c'_j\right)}$ back to messages in $M_j^{c'_j}$.

$$\eta_j^{c'}\left(m_j\right) = \begin{cases}
m_j & \text{if } c'_j = c_j; \\
\phi_k^{c'_k}\left(m_k\right) & \text{if } \left(\begin{array}{c} c'_j \in C_j^F \setminus \{c_j\}, \\ \exists\left[L_j \subset 2^{Y_j} \text{ and } H_j = \{[E_j, y_j] : y_j \in E_j\}\right] \text{ satisfying Definition 4}, \\ \gamma_j\left(c'_j\right)\left[m_j\right] \neq E_j \end{array}\right) \\
\left[E_k, t_k\left[c', m_k\right]\right] & \text{if } \left(\begin{array}{c} c'_j \in C_j^F \setminus \{c_j\}, \\ \exists\left[L_j \subset 2^{Y_j} \text{ and } H_j = \{[E_j, y_j] : y_j \in E_j\}\right] \text{ satisfying Definition 4}, \\ \gamma_j\left(c'_j\right)\left[m_j\right] = E_j \end{array}\right)
\end{cases}$$

Then, define

$$\begin{aligned}
&\bar{b}_j\left(q_j\right)\left[N \times \{\theta\}\right] \equiv b_j\left(\Sigma_j\left(q_j\right)\right)\left[\left\{\left(\left(m_k\right)_{k \in \mathcal{J} \setminus \{j\}}, \theta\right) : \left[\eta_k^{c^{q_j}}\left(m_k\right)\right]_{k \in \mathcal{J} \setminus \{j\}} \in N\right\}\right], \\
&\forall N \subset \times_{k \in \mathcal{J} \setminus \{j\}}\left(M_k^{c^{q_j}}\right), \forall \theta \in \Theta,
\end{aligned}$$

i.e., $\bar{b}_j(q_j)$ mimics $b_j(\Sigma_j(q_j))$ subject to re-labeling messages via $\eta_k^{c^{q_j}}$. Similarly, for each $q_j = (\Gamma_j(\overline{c^{q_j}}), \Psi_j(\overline{m})) \in Q_j^F \diagdown Q_j^{F*}$, $\bar{b}_j(q_j)$ mimics

$$b_j(\Sigma_j(q_j)) = b_j\left(\left[\Gamma_j\left[\gamma\left(\overline{c^{q_j}}\right)\right], \Psi_j\left[\phi^{\overline{c^{q_j}}}(\overline{m})\right]\right]\right).$$

Thus, define

$$\bar{b}_j(q_j)[N \times \{\theta\}] \equiv b_j(\Sigma_j(q_j))\left[\left\{\left((m_k)_{k\in\mathcal{J}\diagdown\{j\}}, \theta\right) : \left[\eta_k^{\overline{c^{q_j}}}(m_k)\right]_{k\in\mathcal{J}\diagdown\{j\}} \in N\right\}\right],$$
$$\forall N \subset \times_{k\in\mathcal{J}\diagdown\{j\}}\left(M_k^{\overline{c^{q_j}}_k}\right), \forall\theta \in \Theta,$$

i.e., $\bar{b}_j(q_j)$ mimics $b_j(\Sigma_j(q_j))$ subject to re-labeling messages via $\eta_k^{\overline{c^{q_j}}}$.

It is straightforward to see that (c, s, t) is replicated by $(c, \bar{s}, \bar{t})$, and all the players inherit incentive compatibility from (c, s, t), i.e., $(c, \bar{s}, \bar{t})$ is a $[C^I, C^F]$-equilibrium and $z^{(c,s,t)} = z^{(c,\bar{s},\bar{t})}$.■

A.4 Proof of Theorem 3

Proof. Fix $\langle\mathcal{A}, \Gamma, \Psi\rangle = \langle\mathcal{A}^{non-delegated}, \Gamma^{private}, \Psi^{private}\rangle$. We have

$$Z^{\mathcal{E}^{\langle\mathcal{A}, \Gamma, \Psi\rangle\text{-}[C^{\mathcal{A}}, C^{\mathcal{A}}]}} \supset Z^{\mathcal{E}^{\langle\mathcal{A}, \Gamma, \Psi\rangle\text{-}[C^{P}, C^{\mathcal{A}}]}}, \tag{28}$$
$$Z^{\mathcal{E}^{\langle\mathcal{A}, \Gamma, \Psi\rangle\text{-}[C^{\mathcal{A}}, C^{\mathcal{A}}]}} \subset Z^{\mathcal{E}^{\langle\mathcal{A}, \Gamma, \Psi\rangle\text{-}[C^{P}, C^{\mathcal{A}}]}}, \tag{29}$$
$$Z^{\mathcal{E}^{\langle\mathcal{A}, \Gamma, \Psi\rangle\text{-}[C^{P}, C^{\mathcal{A}}]}} = Z^{\mathcal{E}^{\langle\mathcal{A}, \Gamma, \Psi\rangle\text{-}[C^{P}, C^{F}]}}, \tag{30}$$

where (28) follows from Lemma 1 and $C^{\mathcal{A}} \sqsupset^* C^P$, (29) from Lemma 3, and (30) from Proposition 2. (28), (29) and (30) imply Theorem 3.■

A.5 Proof of Theorem 4

Proof. Fix $\langle\mathcal{A}, \Gamma, \Psi\rangle = \langle\mathcal{A}^{non-delegated}, \Gamma^{private}, \Psi^{private}\rangle$. We have

$$Z^{\mathcal{E}^{\langle\mathcal{A}, \Gamma, \Psi\rangle\text{-}[C^{\mathcal{A}}, C^{\mathcal{A}}]}} = Z^{\mathcal{E}^{\langle\mathcal{A}, \Gamma, \Psi\rangle\text{-}[C^{P}, C^{F}]}}, \tag{31}$$
$$Z^{\mathcal{E}^{\langle\mathcal{A}, \Gamma, \Psi\rangle\text{-}[C^{P}, C^{P}]}} \supset Z^{\mathcal{E}^{\langle\mathcal{A}, \Gamma, \Psi\rangle\text{-}[C^{P}, C^{F}]}}, \tag{32}$$

where (31) follows from Theorem 3 and (32) from Lemma 2 and $C^P \sqsupset^{**} C^F$. Thus, (31) and (32) imply Theorem 4.■

References

Bernheim, D., and M. Whinston (1985): "Common Marketing Agency as a Device for Facilitating Collusion," *RAND Journal of Economics*, 16, 269–281.

——— (1986a): "Common Agency," *Econometrica*, 54, 923–942.

——— (1986b): "Menu Auctions, Resource Allocations and Economic Influence," *Quarterly Journal of Economics*, 101, 1–31.

Bester, H., and R. Strausz (2000): "Imperfect Commitment and the Revelation Principle: the Multi-agent Case," *Economics Letters*, 69, 165–171.

——— (2001): "Contracting with Imperfect Commitment and the Revelation Principle," *Econometrica*, 69, 1077–1098.

——— (2007): "Contracting with Imperfect Commitment and Noisy Communication," *Journal of Economic Theory*, 136, 236–259.

Chirinko, R. S., and D. J. Wilson (2018): "Tax Competition among U.S. States: Racing to the Bottom or Riding on a Seesaw?," *Journal of Public Economics*, 155, 147–163.

d'Aspremont, C., and R. D. S. Ferreira (2010): "Oligopolistic Competition as a Common Agency Game," *Games and Economic Behavior*, 70, 21–33.

Dixit, A., G. Grossman, and E. Helpman (1997): "Common Agency and Coordination," *Journal of Political Economy*, 105, 752–769.

Doval, L., and V. Skreta (2021): "Mechanism Design with Limited Commitment," *Econometrica, forthcoming.*

Esteller-More, A., U. Galmarini, and L. Rizzo (2012): "Vertical Tax Competition and Consumption Externalities in a Federation with Lobbying," *Journal of Public Economics*, 96, 295–305.

Frandsen, B., and M. Powell (2019): "Sticking Points: Common-Agency Problems and Contracting in the US Healthcare System," *RAND Journal of Economics*, 50, 251–285.

Grossman, G., and E. Helpman (1994): "Protection for Sale," *American Economic Review*, 84, 833–850.

Han, S., and S. Xiong (2022): "Online Appendix to "Common Agency with Non-Delegation or Imperfect Commitment"," *mimeo.*

Keen, M., and K. A. Konrad (2013): *The Theory of International Tax Competition and Coordination.*, vol. Vol. 5 of *Handbook of Public Economics*. Elsevier, Amsterdam & Oxford.

Khalila, F., D. Martimort, and B. Parigic (2007): "Monitoring a Common Agent: Implications for Financial Contracting," *Journal of Economic Theory*, 135, 35–67.

Martimort, D., and A. Semenov (2008): "Ideological Uncertainty and Lobbying Competition," *Journal of Public Economics*, 92, 456–481.

Martimort, D., and L. Stole (2002): "The Revelation and Delegation Principles in Common Agency Games," *Econometrica*, 70, 1659–1673.

McAfee, P. (1993): "Mechanism Design by Competing Sellers," *Econometrica*, 61, 1281–1312.

Myerson, R. (1979): "Incentive Compatibility and the Bargaining Problem," *Econometica*, 47, 61–73.

Parlour, C. A., and U. Rajan (2001): "Competition in Loan Contracts," *American Economic Review*, 91, 1311–1328.

Pavan, A., I. Segal, and J. Toikka (2014): "Dynamic Mechanism Design: A Myersonian Approach," *Econometrica*, 82, 601–653.

Peters, M. (2001): "Common Agency and the Revelation Principle," *Econometrica*, 69, 1349–1372.

——— (2014): "Competing Mechanisms," *Canadian Journal of Economics*, 47, 373–397.

Szentes, B. (2009): "A Note on 'Mechanism Games with Multiple Principals and Three or More Agents' by T. Yamashita," *mimeo*.

Yamashita, T. (2010): "Mechanism Games with Multiple Principals and Three or More Agents," *Econometrica*, 78, 791–801.