Homework Problems on Matrices

Soc. 761

Fall 2014

Most of these problems are adapted from P. J. Davis, *The Mathematics of Matrices (*Xerox College Pub., 1973). Unless otherwise indicated, the problems are meant for hand computation (i.e., using paper and pencil, possibly with the aid of a calculator), but you can check your work in R. Problems marked with an asterisk are optional.

1. Use the double-subscript notation to write down the elements of the second-last column of an $(m \times n)$ matrix $\mathbf{A}$.

2. Use the double-subscript notation to write down the elements of the diagonal that is not the main diagonal of the order-n square matrix $\mathbf{B}$ (i.e., the diagonal elements proceeding from lower left to upper right).

3. Find $\mathbf{A}'$ for

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

   Find $\mathbf{b}$ for $\mathbf{b}' = [2, 1, 6, 9, 4]$.

4. * If A is an order-n square symmetric matrix, show that it can have as many as $\frac{1}{2}n(n+1)$ distinct (i.e., potentially different) elements in it.

5. Given the matrices

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 2 & 0 \\ 7 & -1 \end{bmatrix} \\ \mathbf{B} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ \mathbf{C} &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

   find $\mathbf{A} + \mathbf{I} + \mathbf{0}$ and $\mathbf{A} + \mathbf{B} + \mathbf{C}$, [where $\mathbf{I}$ is the order-2 identity matrix and $\mathbf{0}$ is the (2×2) zero matrix.]

6. If two matrices are symmetric, is their sum necessarily symmetric? Either make up an example to demonstrate your answer, prove it for the (2×2) case, or prove it in general.

7. For the matrices in problem 5, find $\mathbf{A} - \mathbf{B}$.

8. Given the matrices

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix} \\ \mathbf{B} &= \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \end{aligned}$$

   compute $3\mathbf{A} + 2\mathbf{B}$.

9. Compute the inner products of the following pairs of vectors:

$$[-1, 2, 1] \text{ and } \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

$$[0, x, y] \text{ and } [x, x, 3]$$

$$[x, y, z] \text{ and } \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

10. Let $\mathbf{a}_1'$ and $\mathbf{a}_2'$ represent the two *rows* of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

and let $\mathbf{b}_1$ and $\mathbf{b}_2$ represent the two *columns* of the matrix

$$\mathbf{B} = \begin{bmatrix} 4 & -1 \\ 1 & 0 \end{bmatrix}$$

Show how the matrix product $\mathbf{AB}$ is composed of the four inner products $\mathbf{a}_1' \cdot \mathbf{b}_1$, $\mathbf{a}_1' \cdot \mathbf{b}_2$, $\mathbf{a}_2' \cdot \mathbf{b}_1$, and $\mathbf{a}_2' \cdot \mathbf{b}_2$

11. Find the following matrix products:

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$$

$$[2, 0, 5] \begin{bmatrix} 9 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 0 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

12. If both $\mathbf{AB}$ and $\mathbf{BA}$ can be formed, what can be said about the orders of $\mathbf{A}$ and $\mathbf{B}$?

13. * Is the product of two symmetric matrices necessarily symmetric? Support your answer with an example or prove it for the (2×2) case.

14. Using R to do the computations, show that $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$ for the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

15. Express each of the following systems of equations in matrix form:

(a)

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 3 \\ x_1 + 3x_2 + 2x_3 &= 2 \\ 3x_1 + 4x_2 + 3x_3 &= 5 \end{aligned}$$

(b)

$$\begin{aligned} x_1 - x_2 &= 0 \\ 3x_1 - 2x_2 &= 1 \\ 4x_1 - 2x_2 &= 3 \end{aligned}$$

(c)

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_1 + x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 5 \end{aligned}$$

(d)

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 0 \\ x_1 + 3x_2 + 2x_3 &= 0 \end{aligned}$$

16. Using R, find the inverse of each of the following matrices, or determine that the matrix is singular:

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{7}{4} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -6 \\ 3 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 5 \\ 4 & 3 \end{bmatrix}$$

17. Working by hand employing Gaussian elimination, find the inverse of the fourth matrix in problem 16 (or determine that the matrix has no inverse). * Optionally do this for the other three matrices in problem 16.

18. Using R, show by direct computation that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ for

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 1 & 6 \\ 2 & 1 \end{bmatrix} \\ \mathbf{B} &= \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \end{aligned}$$

19. For the system of equation in problem 15 (c) [* and optionally for (a), (b), and (d)], using Gaussian elimination,

    (a) determine the rank of the coefficient matrix;

    (b) decide whether the system of equations is underdetermined, overdetermined, or has a unique solution; and

    (c) if a solution exists, state the solution or solutions.

    (d) Check your work with R.

20. The file `Thurstone.txt` on the course web site contains a correlation matrix among nine psychological tests; these data are a classical example in the literature on factor analysis. If you have an active Internet connection, you can read the data into an R data frame by the following command. (Otherwise, download the file and read it from a local disk.)

```
Thurstone <- read.table("http://socserv.socsci.mcmaster.ca/jfox/Courses/soc761/Thurstone.txt",
                        header=TRUE)
```

Using R, find the eigenvalues and eigenvectors of this correlation matrix. (Note: the `eigen` function will take a numeric data frame as an argument, first "coercing" it to a matrix.)