# Homework Problems on Calculus

## Soc. 761

## Fall 2014

Problems marked with an asterisk are optional. Problems 1–3 are adapted from Kleppner and Ramsey, *Quick Calculus* (Wiley, 1972).

1. * Find the following limits, if they exist:

   (a)
   $$\lim_{x \to 2} \frac{x^2 - 4x + 4}{x - 2}$$

   (b)
   $$\lim_{x \to 0} \frac{x^2 + x + 1}{x}$$

   (c)
   $$\lim_{x \to \infty} \frac{1}{x}$$

2. Find the derivative of each of the following functions of $x$ (where $a$ and $b$ are constants):

   (a)
   $$y = x + x^2 + x^3$$

   (b)
   $$y = (a + bx) + (a + bx)^2 + (a + bx)^3$$

   (c)
   $$y = (3x^2 + 7x)^{-3}$$

   (d)
   $$y = \sqrt{a^2 + x^2}$$

   *Hint*: Recall that $\sqrt{z} = z^{\frac{1}{2}}$.

   (e)
   $$y = \log_e(x^2 + x)$$

   Here $e \approx 2.718$ is a constant (the base of the natural logs).

   (f)
   $$y = e^{x^2 + x}$$

3. * Use the definition of the derivative as the limit of the difference quotient to differentiate the following function with respect to $x$:
   $$y = f(x) = 4x^2 + 4x + 2$$

4. Find the first and second derivatives of each of the following polynomial functions of $x$. For each funciton, determine the points at which the first derivative is 0, and use the second derivative to decide whether each such stationary point is a local minimum, a local maximum, or neither. Determine the value of the function (i.e., the value of $y$) at each point where the first derivative is 0.

(a)
$$y = \frac{x^3}{3} - 16x + 7$$

(b)
$$y = 4x^2 + 4x + 2$$

(c)
$$y = 8 - 6x$$

5. * The problem of finding least-squares regression coefficients for a simple regression can be phrased in the following manner: The equation relating the response variable $y$ to the explanatory variable $x$ for the $i$th of $n$ observations is
$$y_i = a + bx_i + e_i$$

where $a$ and $b$ are the regression coefficients (to be determined), and $e_i$ is the residual for the $i$th observation. Solving this equation for the residual,
$$e_i = y_i - a - bx_i$$

The squared residual for the $i$th observation is therefore
$$e_i^2 = (y_i - a - bx_i)^2$$

The sum of squared residuals over all $n$ observations can be thought of as a function of the regression coefficients $a$ and $b$, since the observed values of the variables $x_i$ and $y_i$ are known constants for a particular sample of observations:
$$S(a, b) = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

The problem is to determine the values of $a$ and $b$ that minimize the sum-of-squares function $S(a, b)$. This problem can be handled by finding the partial derivatives of $S(a, b)$ with respect to $a$ and $b$; setting the partial derivatives to 0; and solving the resulting system of two equations for $a$ and $b$. Derive these equations for $a$ and $b$ without actually solving them. *Hint*: Use the chain rule.

6. Find the vector partial derivatives of the following functions of **x**:

(a)
$$y = [3, 2, 1, 6] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(b)
$$y = \underset{(1 \times n)}{\mathbf{x}'} \underset{}{\mathbf{I}_n} \underset{(n \times 1)}{\mathbf{x}}$$

(c)
$$y = [x_1, x_2] \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

*Hint*: Be careful with this one.

7. For each of the following functions: Find an antiderivative (indefinite integral) of the function; use the antiderivative to compute the area between $x = 0$ and $x = 2$; and * draw a graph of the function showing this area.

(a)
$$y = f(x) = 4x^3 + 3x^2 + 2x + 1$$

(b)
$$y = f(x) = e^x$$

(c) *
$$y = f(x) = \frac{1}{x + 1}$$