Sociology 761 John Fox

Lecture Notes

Very Quick Calculus

1. Introduction

- What is now called *calculus* deals with two basic types of problems: the slopes of tangent lines to curves (*differential calculus*) and areas under curves (*integral calculus*).
- In the 17th century, Newton and Leibnitz independently demonstrated the relationship between these two kinds of problems, consolidating and extending previous work in mathematics dating to the classical period. Newton and Leibnitz are generally acknowledged as the founders of calculus.
- In the 19th century, the concept of limits to functions was introduced to provide a sound logical foundation for calculus.

- We will briefly take up the following topics
  (a) Limits of functions
  (b) The derivative of a function.
  (c) Application of derivatives to optimization problems.
  (d) Partial derivatives of functions of several variables.
  (e) Differential calculus in matrix form.
  (f) Essential ideas of integral calculus.

2. Limits

- Calculus deals with *functions* of the form $y = f(x)$. We will consider the case where both the domain (values of the independent variable x) and range (values of the dependent variable y) of the function are real numbers.
- *Definition of a limit:* A function $y = f(x)$ has a *limit* L at $x = x_0$ (i.e., a particular value of x) if for any positive tolerance ϵ (epsilon), no matter how small, there exists a positive number δ (delta) such that the distance between $f(x)$ and L is less than the tolerance as long as the distance between x and x_0 is smaller than δ — that is, as long as x is confined to a neighbourhood around x_0.
  - In symbols:
  $$|f(x) - L| < \epsilon$$
  for all
  $$0 < |x - x_0| < \delta$$

- This situation is illustrated in Figure 1.
- Note that $f(x_0)$ need not equal L, and need not exist at all. Indeed, limits are often most useful when $f(x)$ does not exist at $x = x_0$.
- The following notation is used:

$$\lim_{x \to x_0} f(x) = L$$

▶ *An example:* Find the limit of

$$y = f(x) = \frac{x^2 - 1}{x - 1}$$

at $x_0 = 1$.

- Note that $f(1) = \frac{1-1}{1-1} = \frac{0}{0}$ is undefined.
- Nevertheless, as long as x is not *exactly* equal to 1, we can divide by $x - 1$:

$$y = \frac{(x+1)(x-1)}{x-1} = x + 1$$

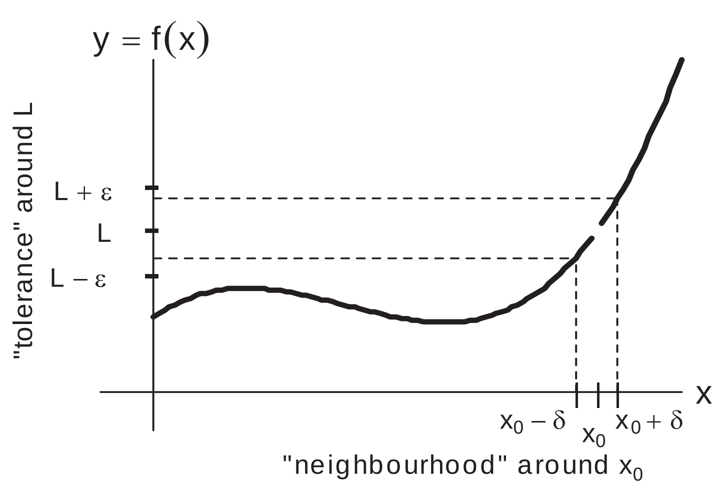

Figure 1. $\lim_{x \to x_0} f(x) = L$

- Since $x_0 + 1 = 1 + 1 = 2$,

$$\begin{aligned}\lim_{x\to 1}\frac{x^2-1}{x-1} &= \lim_{x\to 1}(x+1)\\ &= 1+1=2\end{aligned}$$

- This limit is graphed in Figure 2.

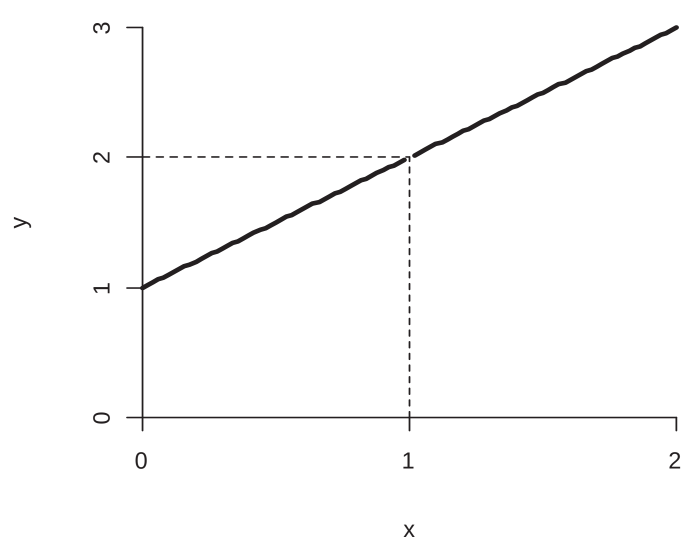

Figure 2. $\lim_{x\to 1}\frac{x^2-1}{x-1} = 2$, even though the function is undefined at $x = 1$.

- Some rules for manipulating limits: If

$$\lim_{x \to x_0} f(x) = a$$

and

$$\lim_{x \to x_0} g(x) = b$$

then

$$\lim_{x \to x_0} [f(x) + g(x)] = a + b$$
$$\lim_{x \to x_0} [f(x)g(x)] = ab$$
$$\lim_{x \to x_0} [f(x)/g(x)] = a/b$$

(the last as long as $b \neq 0$).

3. The Derivative of a Function

- Consider a function $y = f(x)$ evaluated at two values of x:

$$\begin{array}{ll} \text{at } x_1\text{:} & y_1 = f(x_1) \\ \text{at } x_2\text{:} & y_2 = f(x_2) \end{array}$$

  - The *difference quotient* is defined as the change in y divided by the change in x:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

  - As illustrated in Figure 3, the difference quotient is the slope of the line connecting the points (x_1, y_1) and (x_2, y_2).

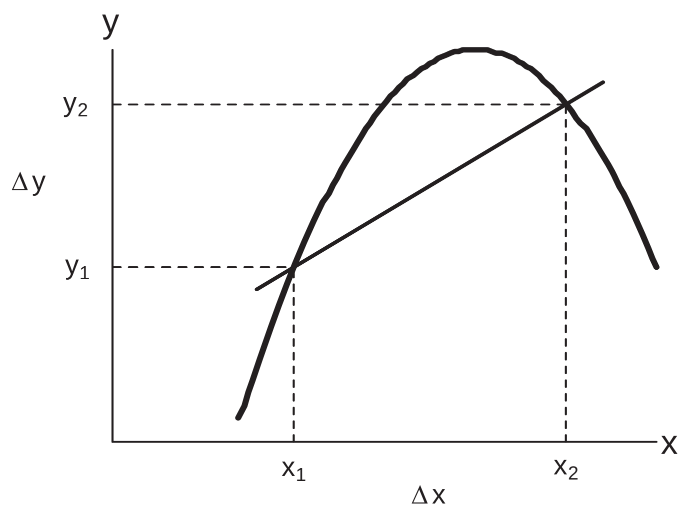

Figure 3. The difference quotient $\Delta y / \Delta x$ is the slope of the line connecting (x_1, y_1) and (x_2, y_2).

- The *derivative* of the function $f(x)$ at $x = x_1$ is the limit of the difference quotient $\Delta y / \Delta x$ as x_2 approaches x_1 (i.e., as $\Delta x \rightarrow 0$):

$$\begin{aligned} \frac{dy}{dx} &= \lim_{x_2 \rightarrow x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \end{aligned}$$

  - The derivative is therefore the slope of the tangent line to the curve $f(x)$ at $x = x_1$.
  - This situation is illustrated in Figure 4.
- Alternative notation for the derivative:

$$\frac{dy}{dx} = \frac{df(x)}{dx} = f'(x)$$

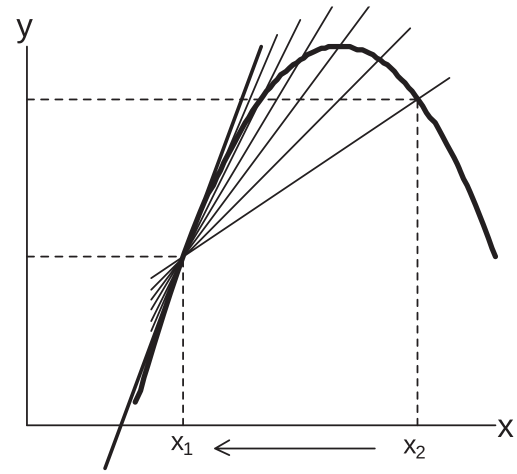

Figure 4. The derivative is the slope of the tangent line at $f(x_1)$.

- The last form, $f'(x)$, emphasizes that the derivative is itself a function of x.

- *An example:* Given the function $y = f(x) = x^2$, find the derivative $f'(x)$ for any value of x.
  - Applying the definition of the derivative as the limit of the difference quotient,

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} (2x + \Delta x) \\ &= \lim_{\Delta x \to 0} 2x + \lim_{\Delta x \to 0} \Delta x \\ &= 2x + 0 = 2x \end{aligned}$$

  - Notice that division by Δx is justified here because although it approaches 0 in the limit, it never is exactly equal to 0.
  - For example, the slope of the curve $y = f(x) = x^2$ at $x = 3$ is $f'(x) = 2x = 2 \times 3 = 6$.
- More generally, by similar reasoning, the derivative of

$$y = f(x) = ax^n$$

is

$$\frac{dy}{dx} = nax^{n-1}$$

  - *Examples:*

$$\begin{aligned} y &= 3x^6 & \frac{dy}{dx} &= 6 \times 3x^{6-1} = 18x^5 \\ y &= \frac{1}{4x^3} = \tfrac{1}{4}x^{-3} & \frac{dy}{dx} &= -3 \times \tfrac{1}{4}x^{-3-1} = -\frac{3}{4x^4} \end{aligned}$$

3.1 Rules for Manipulating Derivatives

▶ The *addition rule*: For

$$\begin{aligned} h(x) &= f(x) + g(x) \\ h'(x) &= f'(x) + g'(x) \end{aligned}$$

- For example:

$$\begin{aligned} y &= 2x^2 + 3x + 4 \\ \frac{dy}{dx} &= 4x + 3 + 0 = 4x + 3 \end{aligned}$$

- We know, therefore, how to differentiate any *polynomial* function (i.e., any weighted sum of powers of x).
- Notice that the derivative of a constant (4 in the last example) is 0.

▶ The *multiplication rule*:

$$\begin{aligned} h(x) &= f(x)g(x) \\ h'(x) &= f(x)g'(x) + f'(x)g(x) \end{aligned}$$

▶ The *division rule*:

$$\begin{aligned} h(x) &= f(x)/g(x) \\ h'(x) &= \frac{g(x)f'(x) - g'(x)f(x)}{[g(x)]^2} \end{aligned}$$

- The *chain rule*: If $y = f(z)$ and $z = g(x)$, then
$$y = f[g(x)] = h(x)$$
and
$$h'(x) = \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$
as if the *differential* dz in the numerator and the denominator can be cancelled.
  - For example, given
$$y = (x^2 + 3x + 6)^5$$
find dy/dx.
    - Let
$$z = g(x) = x^2 + 3x + 6$$
    - Then
$$y = f(z) = z^5$$

- Differentiating,
$$\begin{aligned}\frac{dy}{dz} &= 5z^4 \\ \frac{dz}{dx} &= 2x + 3\end{aligned}$$
- And so, by the chain rule:
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz} \times \frac{dz}{dx} \\ &= 5z^4(2x + 3)\end{aligned}$$
- Finally, substituting for z,
$$\frac{dy}{dx} = 5(x^2 + 3x + 6)^4(2x + 3)$$

3.2 Derivatives of Logs and Exponentials

- The following results are often useful:
  - The derivative of the *log function*:
$$\frac{d\log_e x}{dx} = \frac{1}{x} = x^{-1}$$
Note that $\log_e$ is the natural-log function, that is, log to the base $e \approx 2.718$.
  - The derivative of an *exponential*:
$$\frac{de^x}{dx} = e^x$$
  - The derivative of an exponential function for any constant a (i.e., not necessarily e):
$$\frac{da^x}{dx} = a^x \log_e a$$

3.3 Second-Order and Higher-Order Derivatives

- Because derivatives are themselves functions, they can be differentiated.
  - The *second derivative* is therefore defined as
$$f''(x) = \frac{d^2y}{dx^2} = \frac{df'(x)}{dx}$$
(Notice the alternative notation.)
  - Likewise, the *third derivative* is
$$f'''(x) = \frac{d^3y}{dx^3} = \frac{df''(x)}{dx}$$
  - and so on.

▶ For example, for the function

$$y = f(x) = 5x^4 + 3x^2 + 6$$

we have the derivatives

$$\begin{aligned} f'(x) &= 20x^3 + 6x \\ f''(x) &= 60x^2 + 6 \\ f'''(x) &= 120x \\ f''''(x) &= 120 \\ f'''''(x) &= 0 \end{aligned}$$

4. Optimization

▶ An important application of derivatives, both in statistics and more generally, is to *maximization* and *minimization* problems (e.g., *maximum* likelihood estimation; *least* squares estimation): that is, finding maximum and minimum values of functions.

▶ As illustrated in Figure 5, when a function is at a *relative maximum* or *minimum* (i.e., a value higher than or lower than surrounding values) or at an *absolute* or *global maximum* or *minimum*, the tangent to the function is flat, and hence the function has a derivative of 0 at that point.

- Note, however, that a function can also have a 0 derivative at a point that is neither a minimum nor a maximum, such as at a *point of inflection* — that is, a point where the direction of curvature changes (see Figure 6).

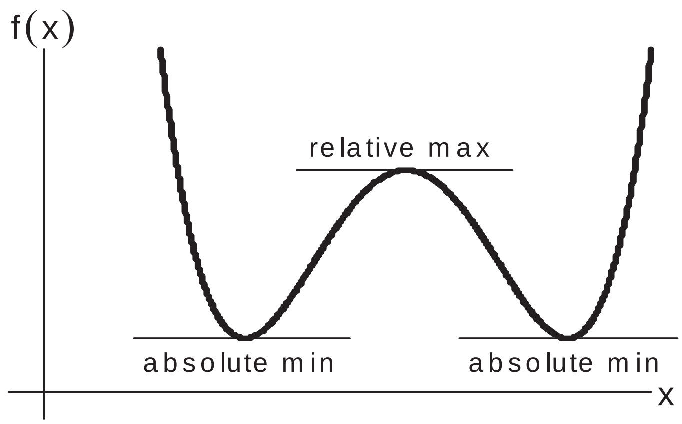

Figure 5. The derivative (i.e., the slope of the function) is 0 at a minimum or maximum.

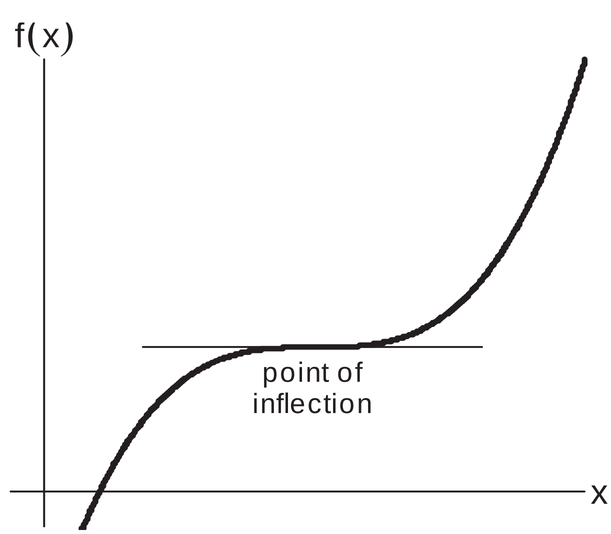

Figure 6. The derivative is also 0 at a point of inflection.

- To distinguish among the three cases — minimum, maximum, or neither — we can appeal to the value of the second derivative (see Figure 7):
  - At a *minimum*, the first derivative $f'(x)$ is changing from negative, through 0, to positive — that is, the first derivative is *increasing*, and therefore the second derivative $f''(x)$ is *positive*.
  - At a *maximum*, the first derivative $f'(x)$ is changing from positive, through 0, to negative — the first derivative is *decreasing*, and therefore the second derivative $f''(x)$ is *negative*.
  - The relationships among the original function, the first derivative, and the second derivative are illustrated in Figure 8

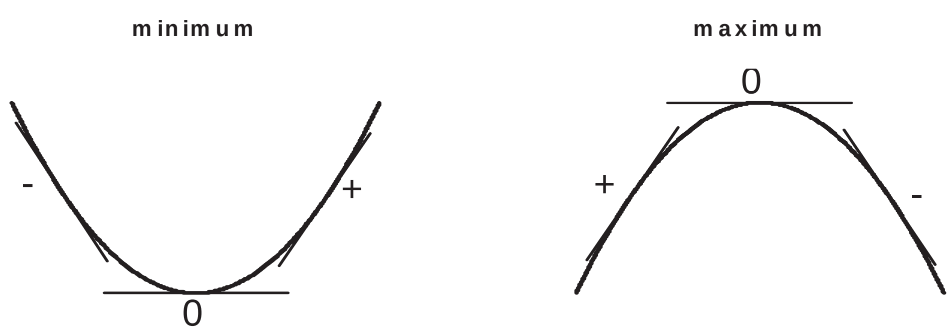

Figure 7. The first derivative is increasing at a minimum and decreasing at a maximum.

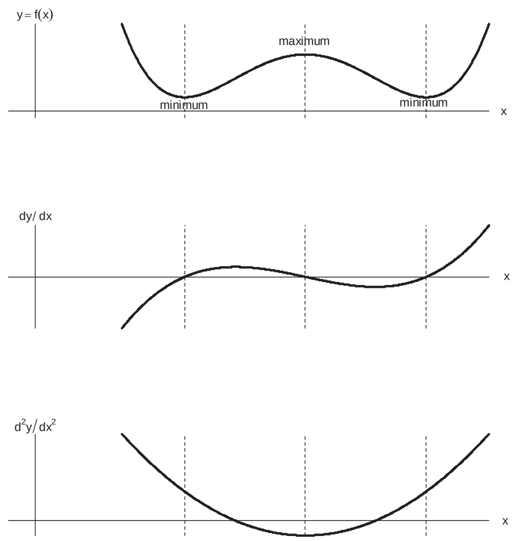

Figure 8. A function and its first and second derivatives.

▶ *An example:* Find extrema (minima and maxima) of the function

$$f(x) = 2x^3 - 9x^2 + 12x + 6$$

- The function is graphed in Figure 9. (By the way: locating *stationary points* — points at which the first derivative is 0 — and determining whether they are minima or maxima, are helpful in graphing functions.)
- The first and second derivatives of the function are

$$\begin{aligned} f'(x) &= 6x^2 - 18x + 12 \\ f''(x) &= 12x - 18 \end{aligned}$$

- Setting the first derivative to 0, and solving for the values of x that satisfy the resulting equation, produces the following results:

$$\begin{aligned} 6x^2 - 18x + 12 &= 0 \\ x^2 - 3x + 2 &= 0 \\ (x-2)(x-1) &= 0 \end{aligned}$$

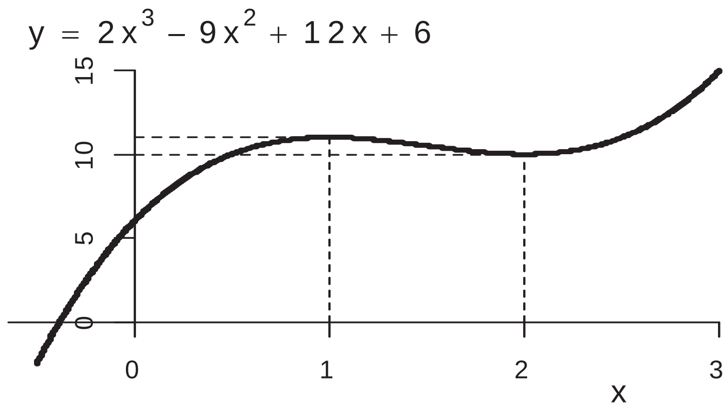

Figure 9. Finding extrema of a function.

and so the roots are

$$x = 2$$
$$x = 1$$

- For $x = 2$:

$$f(2) = 2 \times 2^3 - 9 \times 2^2 + 12 \times 2 + 6 = 10$$
$$f'(2) = 6 \times 2^2 - 18 \times 2 + 12 = 0 \checkmark$$
$$f''(2) = 12 \times 2 - 18 = 6$$

Because $f''(2)$ is positive, the point $(2, 10)$ represents a (relative) minimum.

- Likewise, for $x = 1$:

$$f(1) = 2 \times 1^3 - 9 \times 1^2 + 12 \times 1 + 6 = 11$$
$$f'(1) = 6 \times 1^2 - 18 \times 1 + 12 = 0 \checkmark$$
$$f''(1) = 12 \times 1 - 18 = -6$$

Because $f''(1)$ is negative, the point $(1, 11)$ represents a (relative) maximum.

5. Partial Derivatives

- Now consider a function
$$y = f(x_1, x_2, \ldots, x_n)$$
of several independent variables.
- The *partial derivative* of y with respect to a particular independent variable x_i is the derivative of $f(x_1, x_2, \ldots, x_n)$ treating the other independent variables as constants.
  - That is, the partial derivative indicates how y changes in the direction of x_i holding the other x's constant.
  - *Notation:* The partial derivative of y with respect to x_i is denoted $\dfrac{\partial y}{\partial x_i}$.

▶ For example: The partial derivatives of

$$y = f(x_1, x_2) = x_1^2 + 3x_1x_2 + x_2^3 + 6$$

with respect to x_1 and x_2 are

$$\frac{\partial y}{\partial x_1} = 2x_1 + 3x_2 + 0 + 0 = 2x_1 + 3x_2$$
$$\frac{\partial y}{\partial x_2} = 0 + 3x_1 + 3x_2^2 + 0 = 3x_1 + 3x_2^2$$

▶ To optimize a function of several variables, we can differentiate with respect to each; set the partial derivatives to 0; and solve the resulting set of simultaneous equations.

- When the partial derivatives are linear, we have a system of n linear equations in n unknowns, and the solution is generally straightforward.
- When the partial derivatives are nonlinear, we usually have to use iterative methods (involving successive approximation) to solve the equations.

6. Differential Calculus in Matrix Form

▶ Multi-variable calculus can be simplified by the use of matrices.

▶ Let

$$y = f(x_1, x_2, \ldots, x_n) = f(\underset{(n\times 1)}{\mathbf{x}})$$

- The vector partial derivative of y with respect to the independent-variable vector $\mathbf{x}$ is the vector of partial derivatives of y with respect to the elements of $\mathbf{x}$:

$$\frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{bmatrix}$$

▶ If y is a *linear function* of $\mathbf{x}$,

$$y = \mathbf{a}'\mathbf{x} = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

then $\partial y / \partial x_i = a_i$ and

$$\frac{\partial y}{\partial \mathbf{x}} = \underset{(n \times 1)}{\mathbf{a}}$$

- For example:

$$y = x_1 + 3x_2 - 5x_3 = [1, 3, -5] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}$$

▶ Suppose, now, that y is a *quadratic form* in $\mathbf{x}$:

$$y = \underset{(1 \times n)}{\mathbf{x}'} \underset{(n \times n)}{\mathbf{A}} \underset{(n \times 1)}{\mathbf{x}}$$

where the coefficient matrix $\mathbf{A}$ is symmetric.

- A quadratic form is the matrix analog of a squared term, ax^2, in a scalar expression.
- Note that a quadratic form evaluates to a scalar.
- Expanding the matrix product gives

$$\begin{aligned} y &= a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n \\ &\quad + a_{21}x_2x_1 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n \\ &\quad \vdots \\ &\quad + a_{n1}x_nx_1 + a_{n2}x_nx_2 + \cdots + a_{nn}x_n^2 \\ &= a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 \\ &\quad + 2a_{12}x_1x_2 + \cdots + 2a_{1n}x_1x_n \\ &\quad + \cdots + 2a_{n-1,n}x_{n-1}x_n \end{aligned}$$

- Then differentiating,

$$\frac{\partial y}{\partial x_i} = 2(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ii}x_i + \cdots + a_{in}x_n)$$
$$= 2\mathbf{a}_i'\mathbf{x}$$

where $\mathbf{a}_i'$ is the ith row of the coefficient matrix $\mathbf{A}$.

- Thus,

$$\frac{\partial y}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x}$$

▶ For example, for

$$\begin{aligned} y &= [x_1, x_2] \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= 2x_1^2 + 3x_1x_2 + 3x_2x_1 + x_2^2 \\ &= 2x_1^2 + 6x_1x_2 + x_2^2 \end{aligned}$$

- The partial derivatives are

$$\frac{\partial y}{\partial x_1} = 4x_1 + 6x_2$$
$$\frac{\partial y}{\partial x_2} = 6x_1 + 2x_2$$

- And the vector partial derivative is

$$\begin{aligned} \frac{\partial y}{\partial \mathbf{x}} &= \begin{bmatrix} 4x_1 + 6x_2 \\ 6x_1 + 2x_2 \end{bmatrix} \\ &= 2 \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \checkmark \end{aligned}$$

- Note the strong analogy to scalar derivatives:

$$\frac{dax}{dx} = a \qquad \frac{dax^2}{dx} = 2ax$$
$$\frac{\partial \mathbf{a}'\mathbf{x}}{\partial \mathbf{x}} = \mathbf{a} \qquad \frac{\partial \mathbf{x}'\mathbf{A}\mathbf{x}}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x}$$

- The *Hessian matrix* of second-order partial derivatives of $y = f(\underset{(n\times 1)}{\mathbf{x}})$ is

$$\underset{(n\times n)}{\frac{\partial^2 y}{\partial \mathbf{x}\partial \mathbf{x}'}} = \begin{bmatrix} \frac{\partial^2 y}{\partial x_1^2} & \frac{\partial^2 y}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 y}{\partial x_1 \partial x_n} \\ \frac{\partial^2 y}{\partial x_2 \partial x_1} & \frac{\partial^2 y}{\partial x_2^2} & \cdots & \frac{\partial^2 y}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 y}{\partial x_n \partial x_1} & \frac{\partial^2 y}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 y}{\partial x_n^2} \end{bmatrix}$$

- For example, for the quadratic form $y = \mathbf{x}'\mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is symmetric,

$$\frac{\partial y}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x}$$

$$\frac{\partial^2 y}{\partial \mathbf{x}\partial \mathbf{x}'} = 2\mathbf{A}$$

▶ To optimize a function $y = f(\mathbf{x})$ of several independent variables, find the vector partial derivative of y with respect to $\mathbf{x}$; set this vector derivative to $\mathbf{0}$; and solve the resulting set of simultaneous equations,

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \mathbf{0}$$

for $\mathbf{x}$.

- Suppose that $\mathbf{x} = \mathbf{x}^*$ is a solution of this set of equations. Conditions on the Hessian may be used to help determine whether $f(\mathbf{x}^*)$ is a relative maximum, a relative minimum, of some other kind of stationary point.

7. Essential Ideas of Integral Calculus

7.1 Areas: Definite Integrals

▶ Consider the area A under a curve $f(x)$ between two horizontal coordinates, x_0 and x_1, as illustrated in Figure 10.

- The area can be approximated by dividing the line between x_0 and x_1 into n small intervals, each of width Δx, and constructing a series of rectangles just touching the curve, as shown in Figure 11.
- The x-values defining the rectangles are
$$x_0, x_0 + \Delta x, x_0 + 2\Delta x, \ldots, x_0 + n\Delta x$$
- Consequently the combined area of the rectangles is
$$\sum_{i=0}^{n-1} f(x_0 + i\Delta x)\Delta x \approx A$$

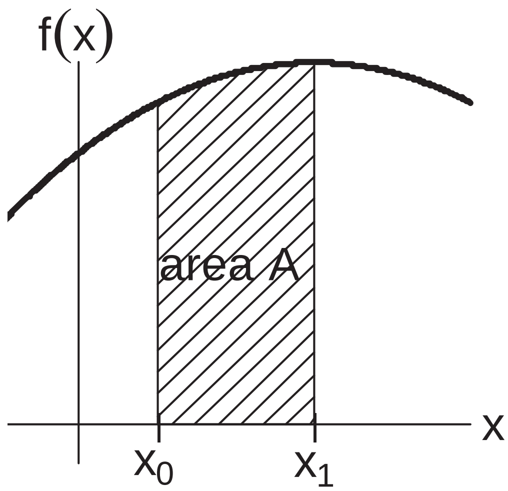

Figure 10. The area A under a function $f(x)$ between x_0 and x_1.

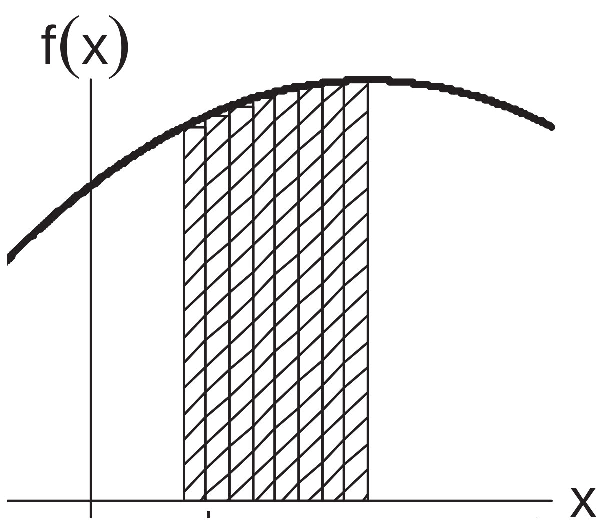

Figure 11. Approximating the area under a curve by summing the areas of rectangles.

- The approximation grows better as the number of rectangles n increases (and Δx grows smaller).
- In the limit,

$$A = \lim_{\substack{\Delta x \to 0 \\ n \to \infty}} \sum_{i=0}^{n-1} f(x_0 + i\Delta x)\Delta x$$

▶ The following notation is used for this limit, which is called the *definite integral* of $f(x)$ from $x = x_0$ to x_1:

$$A = \int_{x_0}^{x_1} f(x)dx$$

▶ The definite integral defines a *signed area*, which may be negative if (some) values of y are less than 0, as illustrated in Figure 12.

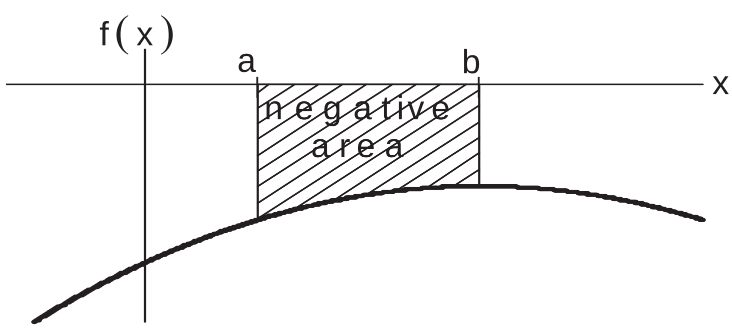

Figure 12. A negative definite integral (area).

7.2 Indefinite Integrals

- Suppose that for the function $f(x)$, there exists a function $F(x)$ such that
$$\frac{dF(x)}{dx} = f(x)$$
- Then $F(x)$ is called an *antiderivative* or *indefinite integral* of $f(x)$.
- The indefinite integral of a function is not unique, for if $F(x)$ is an antiderivative of $f(x)$, then so is $G(x) = F(x) + c$, where c is a constant (i.e., not a function of x).
  - Conversely, if $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, then for some constant c, $G(x) = F(x) + c$.

- For example, for $f(x) = x^3$, the function $\frac{1}{4}x^4 + 10$ is an antiderivative of $f(x)$, as are $\frac{1}{4}x^4 - 10$ and $\frac{1}{4}x^4$.
  - Indeed, any function of the form $F(x) = \frac{1}{4}x^4 + c$ will do.
- The following notation is used for the indefinite integral: If
$$\frac{dF(x)}{dx} = f(x)$$
then we write
$$F(x) = \int f(x)dx$$

7.3 The Fundamental Theorem of Calculus

▶ Newton and Leibnitz figured out the connection between antiderivatives and areas under curves. The relationship they discovered between indefinite and definite integrals is called the *fundamental theorem of calculus*:

$$\int_{x_0}^{x_1} f(x)dx = F(x_1) - F(x_0)$$

where $F(\cdot)$ is *any* antiderivative of $f(\cdot)$.

▶ Here is a sketch of a proof of this theorem:

- Consider the area $A(x)$ under the curve $f(x)$ between some fixed value x_0 and another (moveable) value x, as illustrated in Figure 13.
- In this picture, $x + \Delta x$ is a value slightly to the right of x and ΔA is the area under the curve between x and $x + \Delta x$.
- A rectangular approximation to this small area is

$$\Delta A \approx f(x)\Delta x$$

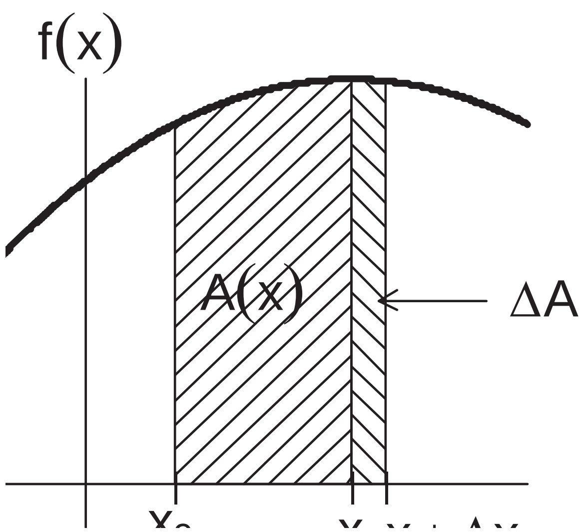

Figure 13. The area $A(x)$ under the curve between the fixed value x_0 and another value x.

- The area ΔA is also

$$\Delta A = A(x + \Delta x) - A(x)$$

- Taking the derivative of A,

$$\begin{aligned} \frac{dA(x)}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(x)\Delta x}{\Delta x} \\ &= f(x) \end{aligned}$$

- Consequently,

$$A(x) = \int f(x)dx$$

a specific, but as yet unknown, indefinite integral of $f(x)$.

- Let $F(x)$ be some other specific, arbitrary, indefinite integral of $f(x)$.
  – Then $A(x) = F(x) + c$, for some c.
- We know that $A(x_0) = 0$ — because $A(x)$ is the area under the curve between x_0 and x.
  – Thus

$$\begin{aligned} A(x_0) &= F(x_0) + c = 0 \\ c &= -F(x_0) \end{aligned}$$

  and for a particular value of $x = x_1$

$$A(x_1) = \int_{x_0}^{x_1} f(x)dx = F(x_1) - F(x_0)$$

  where (recall) $F(\cdot)$ is *an arbitrary* antiderivative of $f(\cdot)$.

▶ For example, let us find the area (evaluate the definite integral)

$$A = \int_1^3 (x^2 + 3)dx$$

- It is convenient to use

$$F(x) = \frac{1}{3}x^3 + 3x$$

[Verify that $F(x)$ is an antiderivative of $f(x) = x^2 + 3$.]

- Then

$$\begin{aligned} A &= F(3) - F(1) \\ &= \left(\frac{1}{3}3^3 + 3 \times 3\right) - \left(\frac{1}{3}1^3 + 3 \times 1\right) \\ &= 18 - 3\tfrac{1}{3} = 14\tfrac{2}{3} \end{aligned}$$