Strongly Robust Equilibrium and Competing-Mechanism Games

Seungjin Han*

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Abstract

This paper formulates the notion of a strongly robust equilibrium relative to a set of mechanisms specified in any competing-mechanism game of complete information with multiple principals and multiple agents. It shows that when agents' efforts are contractible, any strongly robust pure-strategy equilibrium relative to single-incentive contracts persists, regardless of the continuation equilibrium that agents play upon any principal's deviation to any complex mechanism.

Keywords: robust equilibrium, competing-mechanism games, multiple principals and multiple agents

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*Address: Department of Economics, McMaster University 280 Main Street West, Hamilton, Ontario L8S 4M4, Canada. I am very grateful to the Associate Editor and an anonymous referee for their constructive comments and suggestions. I also thank the seminar audience at Yonsei University and the 2006 Far Eastern Meeting of the Econometric Society for their comments. Financial support from SSHRC of Canada is gratefully acknowledged. All errors are mine alone. Phone: +1-905-525-9140 x23818. Fax: +1-905-521-8232. E-mail: hansj@mcmaster.ca.

1 Introduction

When principals (e.g., sellers) compete in the market, agents (e.g., buyers) often have information on what other principals are doing. If principals use complex mechanisms that make their contracts contingent on agents' reports on what other principals are doing, they may actively change their contracts in response to changes in the market. It seems clear that the competition among principals should be modeled by allowing them to use complex mechanisms that ask agents to report their private information not only on their payoff types, but also on what other principals are doing.[1] However, this creates a great deal of difficulty in specifying the message spaces for such mechanisms. For example, a principal may wish to make a contract that depends on mechanisms or contracts that other principals choose, and other principals may also want to make contracts that depend on the mechanisms or contracts of other principals, etc. Therefore, agents may have to report to a principal something that involves an infinite regression that must be resolved. A solution to this infinite regression problem has been proposed by Epstein and Peters [6]. They developed a class of mechanisms with a universal language that allows agents to describe any complex mechanisms that other principals might offer. However, this language itself is quite complex to apply to economic problems observed in practice.

Subsequently, theoretical research on competing principals has focussed on the development of competing-mechanism games that reduce this complexity of communication, yet generate equilibrium allocations relative to any complex mechanisms [7, 9, 13, 15, 21]. However, most applied research on competing principals does not allow communication between principals and agents prior to settling on a contract. For example, the best-known results on competing principals when agents' efforts are contractible are based on equilibria of single-contract games, whereby each principal simultaneously offers a single-incentive contract to agents [3–5, 8, 10, 12, 18]. A single contract is in fact a degenerate mechanism that assigns the same contract, regardless of the messages sent by agents. This absence of communication in determining a contract makes it quite simple to analyze and characterize an equilibrium relative to single contracts.

In general, simple games such as single-contract games may not gener-

[1] In general, the revelation principle for the direct mechanisms defined over only payoff types does not hold in competing-mechanism games, since agents also have information on mechanisms offered by other principals [6, 9, 14].

ate all the interesting equilibria relative to complex mechanisms [17], but a practical and important question is whether an equilibrium relative to single contracts is robust to the possibility that a principal deviates to a complex mechanism. Peters [16] showed that any pure-strategy equilibrium relative to single contracts is weakly robust when multiple principals deal with only one agent (common agency) and the agent's payoff type is public information, in the sense that there always exists a continuation equilibrium that punishes a deviating principal upon deviation to any complex mechanism. A weakly robust equilibrium may, however, disappear in some continuation equilibrium when there are multiple continuation equilibria upon a principal's deviation to a complex mechanism.[2]

This paper investigates a much stronger robustness for the equilibrium of single-contract games of complete information when multiple principals deal with multiple agents (multiple agency). An equilibrium of a single-contract game is said to be strongly robust relative to single contracts if the equilibrium payoff for each principal is no less than his payoffs in any continuation equilibria (including mixed-strategy continuation equilibria) on the equilibrium path and off the equilibrium path following his unilateral deviations to any single contracts. This paper shows that when agents' efforts are contractible, any strongly robust pure-strategy equilibrium relative to single contracts is also strongly robust for any complex mechanisms that assign a contract conditional on agents' reports. In other words, the strong robustness of an equilibrium relative to single contracts is a sufficient condition for the equilibrium to be strongly robust for any complex mechanisms. This implies that a strongly robust pure-strategy equilibrium relative to single contracts does not disappear, no matter what continuation equilibrium agents might play regarding any principal's deviation to any complex mechanism. The paper considers application of the multiple-agency game of Prat and Rustichini [18] with complete information. Any pure-strategy equilibrium of this game is in fact strongly robust regarding any complex mechanisms. The

[2] Most theoretical research on competing-mechanism games focuses on weak robust equilibria. To the best of my knowledge, the only exception is the notion of robustness as formulated by Epstein and Peters [6]. They showed that any truth-telling equilibrium relative to the universal mechanisms persists in any continuation equilibrium that agents might play regarding any principal's deviation to any complex mechanism. However, agents are restricted to use pure communication strategies. It is not known yet whether a truth-telling continuation equilibrium persists even if agents use mixed communication strategies regarding a principal's deviation to any complex mechanism.

results in this paper show that single-contract games can be examined for strongly robust equilibrium allocations in the case of complete information and contractible effort.

2 Competing-Mechanism Games

Throughout this paper, a set is assumed to be a compact set unless otherwise specified. When a measurable structure is necessary, the corresponding Borel σ algebra is used. For a set X, $\Delta(X)$ denotes the set of probability distributions on X. For any $x \in \Delta(X)$, $\mathsf{supp}\ x$ denotes the support of the probability distribution x.

There are two disjoint sets of players. The set of principals is $\mathcal{J} = \{1, \cdots, J\}$. The set of agents is $\mathcal{I} = \{1, \cdots, I\}$. Each agent i expends effort e_i in the set E_i. Let $E = \times_{s=1}^{I} E_s$. Each principal j may take an action $y^j \in Y^j$. Let $Y = \times_{t=1}^{J} Y^t$. The payoff function for agent i is denoted by $u_i : E \times Y \to \mathbb{R}$ and that for principal j by $v^j : E \times Y \to \mathbb{R}$. This paper considers an environment in which agents' efforts are contractible, so each principal j may write an incentive contract $\alpha^j : E \to \Delta(Y^j)$. An incentive contract α^j specifies the (random) action of principal j as a function of all the agents' efforts. Let $\mathcal{A}^j$ be the set of feasible incentive contracts that principal j can write for agents. Given this notation, let $\mathcal{A} = \times_{t=1}^{J} \mathcal{A}^t$. A typical element $\alpha = (\alpha^1, \cdots, \alpha^J)$ in $\mathcal{A}$ is called a collection of incentive contracts that all principals can write for agents.

First, we construct a competing-mechanism game relative to an arbitrary set of mechanisms and then formulate the notion of a strongly robust equilibrium relative to the set of mechanisms specified in the game. A mechanism offered by principal j to agents is a measurable mapping $\gamma^j : \mathcal{C}^j \to \mathcal{A}^j$, where $\mathcal{C}^j = \times_{s=1}^{I} \mathcal{C}_s^j$, with $\mathcal{C}_s^j$ defined as the set of messages that agent s can send to principal j.[3] For any given mechanism γ^j and any array of messages $c^j = (c_1^j, \cdots, c_I^j) \in \mathcal{C}^j$, $\gamma^j(c^j)$ denotes the incentive contract assigned. Let Γ^j be a set of feasible mechanisms that principal j can offer to agents. A typical element $(\gamma^1, \cdots, \gamma^J)$ in $\Gamma = \times_{t=1}^{J} \Gamma^t$ is called a collection of mecha-

[3] We may also consider more general two-way communication mechanisms [11] that allow the principal to send a private recommendation to each agent after agents send messages to the principal, but before agents make their effort choices. It is not fully known yet how these two-way mechanisms change the nature of competition and the set of equilibrium allocations in a multiple-agency situation.

nisms that all principals can offer to all agents. The mechanisms in Γ can be simple or quite complex in terms of the degree and nature of communication that the mechanisms allow. The competing-mechanism game relative to Γ begins when each principal j simultaneously offers a mechanism from Γ^j. After seeing the collection of mechanisms offered by principals, each agent simultaneously sends a message to each principal and chooses her effort. Subsequently, all payoffs are realized.

The continuation strategy of agent i is a measurable mapping $m_i : \Gamma \rightarrow \Delta\left(\mathcal{C}_i \times E_i\right)$, where $\mathcal{C}_i = \times_{t=1}^{J} \mathcal{C}_i^t$. With a slight abuse of notation, $m_i(\cdot, \cdot|\gamma)$ is the probability distribution on $\mathcal{C}_i \times E_i$ that agent i uses when $\gamma \in \Gamma$ is the collection of mechanisms that principals offer to agents. A profile of continuation strategies $\widehat{m} = \{\widehat{m}_s\}_{s=1}^{I}$ is a continuation equilibrium relative to Γ if for every $i \in \mathcal{I}$ and every $\gamma \in \Gamma$, the continuation strategy $\widehat{m}_i$ maximizes:

$$\int_{\mathcal{C}_{-i} \times E_{-i}} \left(\int_Y u_i(e, y) d\gamma(c)(e) \right) d\widehat{m}_{-i}(c_{-i}, e_{-i}|\gamma),$$

where $\mathcal{C}_{-i} = \times_{s \neq i} \mathcal{C}_s$, $E_{-i} = \times_{s \neq i} E_s$, $\gamma(c)(e) = \Pi_{t=1}^{J} \gamma^t(c^t)(e)$, $e = (e_1, \cdots, e_I)$ and $\widehat{m}_{-i}(c_{-i}, e_{-i}|\gamma) = \Pi_{s \neq i} \widehat{m}_s(c_s, e_s|\gamma)$. Given a continuation equilibrium $\widehat{m} = \{\widehat{m}_s\}_{s=1}^{I}$, principal j chooses a strategy σ^j from $\Delta(\Gamma^j)$. Suppose that the other principals' strategies are $\sigma^{-j} = \{\sigma^t\}_{t \neq j}$. The payoff for principal j is:

$$V^j(\sigma^j, \sigma^{-j}, \widehat{m}) =$$
$$\int_{\Gamma^j} \left\{ \int_{\Gamma^{-j}} \left[\int_{\mathcal{C} \times E} \left(\int_Y v^j(e, y) d\gamma(c)(e) \right) d\widehat{m}(c, e|\gamma) \right] d\sigma^{-j}(\gamma^{-j}) \right\} d\sigma^j(\gamma^j),$$

where $\widehat{m}(c, e|\gamma) = \Pi_{s=1}^{I} \widehat{m}_s(c_s, e_s|\gamma)$ and $\sigma^{-j}(\gamma^{-j}) = \Pi_{t \neq j} \sigma^t(\gamma^t)$. $\{\widehat{\sigma}, \widehat{m}\}$ is an equilibrium relative to Γ if $\widehat{\sigma} = \{\widehat{\sigma}^t\}_{t=1}^{J}$ is a Nash equilibrium of the normal-form game defined by the continuation equilibrium $\widehat{m}$ relative to Γ.

We now formulate the notion of the strong robustness of an equilibrium relative to a set of mechanisms Γ specified in a competing-mechanism game.

Definition 1 *An equilibrium $\{\widehat{\sigma}, \widehat{m}\}$ is strongly robust relative to Γ if for all $j \in \mathcal{J}$ and all $\gamma^j \in \Gamma^j$ and any continuation equilibrium $\widetilde{m}$ relative to Γ, we have:*

$$V^j(\widehat{\sigma}^j, \widehat{\sigma}^{-j}, \widehat{m}) \geq V^j(\gamma^j, \widehat{\sigma}^{-j}, \widetilde{m}).$$

An equilibrium relative to Γ is said to be strongly robust relative to Γ if the equilibrium payoff for each principal is not less than his payoffs in any continuation equilibria (including mixed-strategy continuation equilibria) on the equilibrium path and off the equilibrium path following his unilateral deviations to any mechanisms in Γ. Not every equilibrium of a competing-mechanism game is a strongly robust equilibrium. In some cases, a strongly robust equilibrium may not exist at all.

3 A Single-Contract Game

The games most often used in the literature are single-contract games in which each principal offers a single contract to agents and then agents choose efforts given the contracts offered. In our notation, each principal j offers a single contract from $\mathcal{A}^j$. A single contract α^j is a degenerate mechanism $\gamma^j : \mathcal{C}^j \to \mathcal{A}^j$ such that $\gamma^j(c^j) = \alpha^j$ for all $c^j \in \mathcal{C}^j$. A principal assigns the same contract regardless of the messages agents send, so communication has no role in determining the contract offer. This absence of communication makes it quite simple to analyze and characterize an equilibrium relative to single contracts. Single-contract games may not generate all the equilibria relative to complex mechanisms. However, it is important to ask whether equilibria in single-contract games are robust to a principal's deviation to a complex mechanism. This paper employs the notion of strong robustness and asks whether an equilibrium in single-contract games survives in any continuation equilibrium upon any principal's deviation to any complex mechanism.

Let $q_i : \mathcal{A} \to \Delta(E_i)$ denote an effort strategy for agent i and $\tau^j \in \Delta(\mathcal{A}^j)$ a strategy for principal j. A profile of strategies $\{\widetilde{\tau}, \widetilde{q}\}$ is an equilibrium relative to $\mathcal{A}$ if $\widetilde{q} = \{\widetilde{q}_s\}_{s=1}^{I}$ is a continuation equilibrium relative to $\mathcal{A}$ and $\widetilde{\tau} = \{\widetilde{\tau}^t\}_{t=1}^{J}$ is a Nash equilibrium of the normal-form game defined by $\widetilde{q}$. Before presenting the main result of the paper, Theorem 1, an example is introduced below to highlight the key intuition behind Theorem 1.

3.1 Example

There are two principals and one agent. Let $\{y^j, \acute{y}^j\}$ be the set of actions for each principal j. The single agent uses an effort from the set $\{e, e'\}$. Payoffs for principal 1, principal 2 and the agent are listed in Table 1.

	y^2		$\acute{y}^2$	
	e	e'	e	e'
y^1	$2,4,8$	$1,2,3$	$2,3,1$	$3,6,7$
$\acute{y}^1$	$3,3,5$	$2,6,6$	$5,2,4$	$2,3,2$

Table 1 Payoffs

Let $\alpha^j : \{e, e'\} \to [0,1]$ be a single contract for principal j that specifies the probability of action y^j as a function of the agent's effort. When the agent chooses e, principal j takes y^j with probability $\alpha^j(e)$ and $\acute{y}^j$ with probability $1 - \alpha^j(e)$. When the agent chooses effort e', principal j takes y^j with probability $\alpha^j(e')$ and $\acute{y}^j$ with probability $1 - \alpha^j(e')$. Let $\mathcal{A}^j$ be the set of all possible single contracts for principal j. Consider the single-contract game relative to $\mathcal{A} = \mathcal{A}^1 \times \mathcal{A}^2$. In this game, each principal j simultaneously offers a single contract from $\mathcal{A}^j$, the agent chooses her effort, and the action of each principal j is realized given his contract and the agent's effort choice.

In general, an equilibrium relative to $\mathcal{A}$ may not be strongly robust relative to $\mathcal{A}$. Such an equilibrium is not strongly robust to complex mechanisms because a single contract is a degenerate mechanism that assigns the same contract regardless of the message sent by the agent. Therefore, an equilibrium relative to $\mathcal{A}$ must be strongly robust relative to $\mathcal{A}$ if it is strongly robust relative to any complex mechanisms. However, not every strongly robust equilibrium relative to $\mathcal{A}$ is strongly robust relative to any complex mechanisms. To demonstrate this point, we first identify strongly robust equilibrium strategies for principals relative to $\mathcal{A}$ in Claim 1.

When the agent chooses effort e, $\acute{y}^1$ is the strictly dominant action for principal 1 and similarly y^2 is the strictly dominant action for principal 2. Without loss of generality, we can focus on contracts such that $\alpha^1(e) = 0$ for principal 1 and contracts such that $\alpha^2(e) = 1$ for principal 2. Consider two possible contracts, $\hat{\alpha}^1$ and $\ddot{\alpha}^1$, for principal 1 such that $\hat{\alpha}^1$ satisfies $\hat{\alpha}^1(e) = \hat{\alpha}^1(e') = 0$ and $\ddot{\alpha}^1$ satisfies $\ddot{\alpha}^1(e) = 0$ and $\ddot{\alpha}^1(e') = 1$.[4] Suppose that principal 1 offers $\hat{\alpha}^1$ with probability p and $\ddot{\alpha}^1$ with probability $1 - p$: the strategy of principal 1 is therefore denoted by $\widetilde{\tau}^1$ such that $\widetilde{\tau}^1(\hat{\alpha}^1) = p$ and $\widetilde{\tau}^1(\ddot{\alpha}^1) = 1 - p$. Suppose that principal 2 offers $\bar{\alpha}^2$ for sure, where $\bar{\alpha}^2$ satisfies $\bar{\alpha}^2(e) = 1$ and

[4] When principal 1 offers $\hat{\alpha}^1$, he chooses $\acute{y}^1$ for sure, regardless of the agent's effort choice. When principal 1 offers $\ddot{\alpha}^1$, he chooses $\acute{y}^1$ if the agent chooses e and y^1 if the agent chooses e'.

$\bar{\alpha}^2(e') = 0$. The strategy of principal 2 strategy is denoted by $\widetilde{\tau}^2$ such that $\widetilde{\tau}^2(\bar{\alpha}^2) = 1$.

Claim 1 *$\{\widetilde{\tau}^1, \widetilde{\tau}^2\}$ is the profile of strongly robust equilibrium strategies for principals relative to $\mathcal{A}$ for any $p \in [0, 1/2]$.*

The proof of Claim 1 consists of two parts. The first part investigates all continuation equilibria (i.e., all possible optimal effort choices for the agent) on the equilibrium path and the second part considers all possible continuation equilibria off the equilibrium path following the unilateral deviation of each principal j to any contract in $\mathcal{A}^j$. Let $\widetilde{q}(\alpha^1, \alpha^2)$ be the optimal probability with which the agent chooses e at $(\alpha^1, \alpha^2) \in \mathcal{A}$, so it characterizes a continuation equilibrium at (α^1, α^2). First, consider a continuation equilibrium on the equilibrium path. Suppose that principal 1 has drawn $\hat{\alpha}^1$ given his randomization. Then the agent will make her effort choice given contract offers $(\hat{\alpha}^1, \bar{\alpha}^2)$. If the agent chooses e, then principal 1 chooses $\acute{y}^1$ and principal 2 chooses y^2, so the agent's payoff is 5. If the agent chooses e', then principal 1 chooses $\acute{y}^1$, but principal 2 now chooses $\acute{y}^2$, so the agent's payoff is 2. Therefore, $\widetilde{q}(\hat{\alpha}^1, \bar{\alpha}^2) = 1$ is the unique continuation equilibrium at $(\hat{\alpha}^1, \bar{\alpha}^2)$. Subsequently, each principal's payoff is 3. Suppose that the contracts are now $\left(\ddot{\alpha}^1, \bar{\alpha}^2\right)$. If the agent chooses e, then principal 1 chooses $\acute{y}^1$ and principal 2 chooses y^2, so the agent's payoff is 5. If the agent chooses e', then principal 1 chooses y^1, but principal 2 chooses $\acute{y}^2$, so the agent's payoff is 7. Therefore, $\widetilde{q}(\ddot{\alpha}^1, \bar{\alpha}^2) = 0$ is the unique continuation equilibrium at $(\ddot{\alpha}^1, \bar{\alpha}^2)$. Subsequently, principal 1's payoff is 3 and principal 2's payoff is 6. Given $(\widetilde{\tau}^1, \widetilde{\tau}^2)$, the principals' payoffs are:

$$V^1(\widetilde{\tau}^1, \widetilde{\tau}^2, \widetilde{q}) = 3$$
$$V^2(\widetilde{\tau}^1, \widetilde{\tau}^2, \widetilde{q}) = 3p + 6(1-p) = 6 - 3p.$$

Since there exists a unique continuation equilibrium on the equilibrium path, it is sufficient for proof of Claim 1 to show that there is no continuation equilibrium upon the deviation of any principal j to any contract in $\mathcal{A}^j$ that makes the deviating principal strictly better off. This is shown in the Appendix.

Now we show that $\{\widetilde{\tau}^1, \widetilde{\tau}^2\}$ is not strongly robust relative to complex mechanisms if $p \in (0, 1/2]$, even though it is strongly robust relative to $\mathcal{A}$. Given principal 1's randomization, suppose that principal 2 deviates to the mechanism $\bar{\gamma}^2 : \{c, c'\} \to \mathcal{A}$ that satisfies $\bar{\gamma}^2(c) = \breve{\alpha}^2$ and $\bar{\gamma}^2(c') = \bar{\alpha}^2$, where $\breve{\alpha}^2$ is defined by $\breve{\alpha}^2(e) = \breve{\alpha}^2(e') = 1$. Therefore, principal 2 assigns contract $\breve{\alpha}^2$ when the agent sends message c and $\bar{\alpha}^2$ when the agent sends message c'. Suppose that principal 1 has drawn $\hat{\alpha}^1$. If the agent sends message c, then principal 2 assigns $\breve{\alpha}^2$. Since principal 1 always chooses $\acute{y}^1$ under $\hat{\alpha}^1$ and principal 2 always chooses y^2 under $\breve{\alpha}^2$, the agent's unique choice of optimal effort after sending message c is to choose e' for sure and the payoffs for principal 1, principal 2 and the agent are 2, 6 and 6, respectively. In this way, we can derive: (i) the agent's optimal effort choice for every possible pair of principal 1's contract realized from $\widetilde{\tau}^1$ and the message that the agent can send to principal 2; and subsequently (ii) the corresponding payoffs for principal 1, principal 2, and the agent. These payoffs are listed in Table 2.

	c	c'
$\hat{\alpha}^1$	$2, 6, 6$	$3, 3, 5$
$\ddot{\alpha}^1$	$3, 3, 5$	$3, 6, 7$

Table 2 Payoffs upon deviation of principal 2 to $\bar{\gamma}^2$

From Table 2, the unique continuation equilibrium upon principal 2's deviation to $\bar{\gamma}^2$ is that the agent sends message c (i.e., choose contract $\breve{\alpha}^2$) and chooses effort e' when principal 1's contract is $\hat{\alpha}^1$ and sends message c' (i.e., choose contract $\bar{\alpha}^2$) and chooses effort e' when principal 1's contract is $\ddot{\alpha}^1$. Given the unique continuation equilibrium, principal 2's payoff is always 6, no matter what contract principal 1 offers from his randomization. Since $p \in (0, 1/2]$, principal 2's original payoff is $V^2(\widetilde{\tau}^1, \widetilde{\tau}^2, \widetilde{q}) = 6 - 3p < 6$. Therefore, principal 2 gains if he deviates to $\bar{\gamma}^2$. Therefore, even if $\{\widetilde{\tau}^1, \widetilde{\tau}^2\}$ with any $p \in (0, 1/2]$ is a profile of strongly robust equilibrium strategies for principals relative to $\mathcal{A}$, it is not strongly robust relative to complex mechanisms. The key intuition behind this result is that a principal may make his contract offer contingent on the contracts of other principals using a complex mechanism when other principals use mixed strategies for their contract offers. Such a strategic response is not possible when a principal is restricted to offering a single contract. This implies that a mixed-strategy equilibrium relative to $\mathcal{A}$ may not be strongly robust relative to complex mechanisms even if it is relative to $\mathcal{A}$.

Suppose that $p = 0$. This implies that $\{\widetilde{\tau}^1, \widetilde{\tau}^2\}$ is a profile of strongly robust equilibrium pure strategies for principals relative to $\mathcal{A}$. Suppose that principal j deviates to some complex mechanism, say $\gamma^j : \mathcal{C} \to \mathcal{A}$, while the other principal continues to offer $\widetilde{\alpha}^{-j}$, where $\widetilde{\alpha}^{-j} = \ddot{\alpha}^1$ if $j = 2$ and $\widetilde{\alpha}^{-j} = \bar{\alpha}^2$ if $j = 1$. Let a continuation equilibrium at $(\gamma^j, \widetilde{\alpha}^{-j})$ be characterized by a probability distribution $p(\cdot, \cdot|\gamma^j, \widetilde{\alpha}^{-j})$ on $\mathcal{C} \times \{e, e'\}$. Note that $p(\cdot, \cdot|\gamma^j, \widetilde{\alpha}^{-j})$ can be decomposed as $p(c, \tilde{e}|\gamma^j, \widetilde{\alpha}^{-j}) = p^c(c|\tilde{e}, \gamma^j, \widetilde{\alpha}^{-j}) \times p^e(\tilde{e}|\gamma^j, \widetilde{\alpha}^{-j})$ for any $(c, \tilde{e})$, where $\widetilde{p}^c(\cdot|\tilde{e}, \gamma^j, \widetilde{\alpha}^{-j})$ is the probability distribution on messages that agent i uses if she chooses $\tilde{e}$ as her effort at $(\gamma^j, \widetilde{\alpha}^{-j})$, and $\widetilde{p}^e(\cdot|\gamma^j, \widetilde{\alpha}^{-j})$ is the probability distribution that agent i uses for the effort choice. Then, $p^e(\cdot|\gamma^j, \widetilde{\alpha}^{-j})$ is a continuation equilibrium at $(\acute{\alpha}^j, \widetilde{\alpha}^{-j})$, where $\acute{\alpha}^j$ is defined as $\acute{\alpha}^j(e) = \int \gamma^j(c) dp^c(c|e, \gamma^j, \widetilde{\alpha}^{-j})$ and $\acute{\alpha}^j(e') = \int \gamma^j(c) dp^c(c|e', \gamma^j, \widetilde{\alpha}^{-j})$ because the agent's effort choice must be optimal given her message choice. This argument implies that for any continuation equilibrium upon the deviation of any principal j to any complex mechanism, there exists a continuation equilibrium at the corresponding single contract in $\mathcal{A}^j$ that preserves all players' payoffs in the continuation equilibrium upon deviation of principal j to the complex mechanism. Therefore, if the original equilibrium is strongly robust relative to $\mathcal{A}$, then it is strongly robust relative to any complex mechanism. We can conclude that $\{\widetilde{\tau}^1, \widetilde{\tau}^2\}$ with $p = 0$ is the profile of strongly robust equilibrium strategies for principals relative to any complex mechanisms in this example.

3.2 Main Result

Theorem 1 formalizes the intuition behind the result of the example in the multiple-agency case.

Theorem 1 *Any strongly robust pure-strategy equilibrium* $\{\widetilde{\alpha}, \widetilde{q}\}$ *relative to* $\mathcal{A}$ *is a strongly robust equilibrium relative to any* Γ.

Proof. First, note that the equilibrium contract $\widetilde{\alpha}^j$ is the degenerate mechanism in Γ^j that always assigns $\widetilde{\alpha}^j$, regardless of messages sent by agents. Since $\{\widetilde{\alpha}, \widetilde{q}\}$ is strongly robust relative to $\mathcal{A}$, the equilibrium payoff for each principal is no less than his payoffs in any continuation equilibria on the equilibrium path in the game relative to Γ. Therefore, we only need to prove that there is no continuation equilibrium upon his deviation to any mechanism in Γ^j that gives deviating principal j a higher payoff than his equilibrium payoff. Suppose that principal j unilaterally deviates to some mechanism $\gamma^j : \mathcal{C}^j \to \mathcal{A}^j$ in Γ^j. Let $\widetilde{p} = \{\widetilde{p}_s\}_{s=1}^I$ characterize an arbitrary continuation equilibrium at $(\gamma^j, \widetilde{\alpha}^{-j})$. Let $\widetilde{p}_i(\cdot, \cdot|\gamma^j, \widetilde{\alpha}^{-j}) \in \Delta(\mathcal{C}_i^j \times E_i)$ denote the equilibrium probability distribution that agent i uses for communication with principal j and for the effort choice at $(\gamma^j, \widetilde{\alpha}^{-j})$. We rewrite $\widetilde{p}_i(\cdot,\cdot|\gamma^j, \widetilde{\alpha}^{-j})$ as:

$$\widetilde{p}_i(c_i^j, e_i|\gamma^j, \widetilde{\alpha}^{-j}) = \widetilde{p}_i^c(c_i^j|e_i, \gamma^j, \widetilde{\alpha}^{-j}) \times \widetilde{p}_i^e(e_i|\gamma^j, \widetilde{\alpha}^{-j})$$

for all $(c_i^j, e_i) \in \mathcal{C}_i^j \times E_i$, where $\widetilde{p}_i^c(\cdot|e_i, \gamma^j, \widetilde{\alpha}^{-j})$ is the equilibrium probability distribution for messages that agent i uses if she chooses e_i as her effort at $(\gamma^j, \widetilde{\alpha}^{-j})$, and $\widetilde{p}_i^e(\cdot|\gamma^j, \widetilde{\alpha}^{-j})$ is the equilibrium probability distribution for the effort choice of agent i. For simplicity, the following notations are defined:

$$\widetilde{u}_i(e, y^j) = \int_{Y^{-j}} u_i(e, y^j, y^{-j}) d\widetilde{\alpha}^{-j}(e) \quad \text{for all } (e, y^j) \in E \times Y^j$$

$$\acute{\alpha}^j(e) = \int_{\mathcal{C}^j} \gamma^j(c^j)(e) d\widetilde{p}^c(c^j|e, \gamma^j, \widetilde{\alpha}^{-j}) \quad \text{for all } e \in E$$

$$\widetilde{p}^c(c^j|e, \gamma^j, \widetilde{\alpha}^{-j}) = \Pi_{s=1}^I \widetilde{p}_s^c(c_s^j|e_s, \gamma^j, \widetilde{\alpha}^{-j}) \quad \text{for all } (c^j, e) \in \mathcal{C}^j \times E$$

$$\widetilde{p}^e(e|\gamma^j, \widetilde{\alpha}^{-j}) = \Pi_{s=1}^I \widetilde{p}_s^e(e_s|\gamma^j, \widetilde{\alpha}^{-j}) \quad \text{for all } e \in E.$$

Then the payoff for agent i becomes:

$$U_i(\widetilde{p}_i, \widetilde{p}_{-i}, \gamma^j, \widetilde{\alpha}^{-j}) = \tag{1}$$

$$\int_E \left\{ \int_{\mathcal{C}^j} \left[\int_{Y^j} \widetilde{u}_i(e, y^j) d\gamma^j(c^j)(e) \right] d\widetilde{p}^c(c^j|e, \gamma^j, \widetilde{\alpha}^{-j}) \right\} d\widetilde{p}^e(e|\gamma^j, \widetilde{\alpha}^{-j}) =$$

$$\int_E \left\{ \int_{Y^j} \widetilde{u}_i(e, y^j) d \left[\int_{\mathcal{C}^j} \gamma^j(c^j)(e) d\widetilde{p}^c(c^j|e, \gamma^j, \widetilde{\alpha}^{-j}) \right] \right\} d\widetilde{p}^e(e|\gamma^j, \widetilde{\alpha}^{-j}) =$$

$$\int_E \left[\int_{Y^j} \left(\int_{Y^{-j}} u_i(e, y) d\widetilde{\alpha}^{-j}(e) \right) d\acute{\alpha}^j(e) \right] d\widetilde{p}^e(e|\gamma^j, \widetilde{\alpha}^{-j}).$$

The first equality simply shows the value of $U_i(\widetilde{p}_i, \widetilde{p}_{-i}, \gamma^j, \widetilde{\alpha}^{-j})$. Note the second equality in (1). Since $\widetilde{p}^c(\cdot|e, \gamma^j, \widetilde{\alpha}^{-j})$ governs what probability distribution $\gamma^j(c^j)(e)$ is realized for any given $(e, \gamma^j, \widetilde{\alpha}^{-j})$, $\widetilde{p}^c(\cdot|e, \gamma^j, \widetilde{\alpha}^{-j})$ is a lottery over lotteries for any given $(e, \gamma^j, \widetilde{\alpha}^{-j})$. The second equality in (1) holds because agent i is an expected-payoff maximizer, so she is indifferent between the lottery $\widetilde{p}^c(\cdot|e, \gamma^j, \widetilde{\alpha}^{-j})$ over lotteries and the reduced lottery $\int_{\mathcal{C}^j} \gamma^j(c^j)(e) d\widetilde{p}^c(c^j|e, \gamma^j, \widetilde{\alpha}^{-j})$ for any given $(e, \gamma^j, \widetilde{\alpha}^{-j})$. The third equality comes from the definitions of $\widetilde{u}_i(e, y^j)$ and $\acute{\alpha}^j(e)$.

Recall that $\{\widetilde{p}_s\}_{s=1}^I$ characterizes a continuation equilibrium at $(\gamma^j, \widetilde{\alpha}^{-j})$. Therefore, for the communication decision $\widetilde{p}_i^c$ and the profile of the other agent's strategies $\{\widetilde{p}_s\}_{s\neq i}$, the effort decision $\widetilde{p}_i^e$ of agent i must satisfy, for any $e_i \in \mathsf{supp}\ \widetilde{p}_i^e(\cdot|\gamma^j, \widetilde{\alpha}^{-j})$ and all $\acute{e}_i \in E_i$:

$$\int_E \left[\int_{Y^j} \left(\int_{Y^{-j}} u_i(e_i, e_{-i}, y) d\widetilde{\alpha}^{-j}(e_i, e_{-i})\right) d\acute{\alpha}^j(e_i, e_{-i})\right] d\widetilde{p}_{-i}^e(e_{-i}|\gamma^j, \widetilde{\alpha}^{-j}) \geq$$
$$\int_E \left[\int_{Y^j} \left(\int_{Y^{-j}} u_i(\acute{e}_i, e_{-i}, y) d\widetilde{\alpha}^{-j}(\acute{e}_i, e_{-i})\right) d\acute{\alpha}^j(\acute{e}_i, e_{-i})\right] d\widetilde{p}_{-i}^e(e_{-i}|\gamma^j, \widetilde{\alpha}^{-j}).$$

Therefore, $\{\widetilde{p}_s^e\}_{s=1}^I$ characterizes a continuation equilibrium at $(\acute{\alpha}^j, \widetilde{\alpha}^{-j})$.

Since principal j is also an expected-payoff maximizer, his payoff upon deviation to γ^j becomes:

$$V^j(\gamma^j, \widetilde{\alpha}^{-j}, \widetilde{p}) =$$
$$\int_E \left[\int_{Y^j} \left(\int_{Y^{-j}} v^j(e, y) d\widetilde{\alpha}^{-j}(e)\right) d\acute{\alpha}^j(e)\right] d\widetilde{p}^e(e|\gamma^j, \widetilde{\alpha}^{-j}) =$$
$$V^j(\acute{\alpha}^j, \widetilde{\alpha}^{-j}, \widetilde{p}^e).$$

We have $V^j(\widetilde{\alpha}^j, \widetilde{\alpha}^{-j}, \widetilde{q}) \geq V^j(\acute{\alpha}^j, \widetilde{\alpha}^{-j}, \widetilde{p}^e)$ because $\{\widetilde{\alpha}, \widetilde{q}\}$ is a strongly robust equilibrium relative to $\mathcal{A}$. Therefore, it is not profitable for principal j to deviate to any mechanism in Γ^j, regardless of what continuation equilibrium occurs upon deviation. Thus, any strongly robust pure-strategy equilibrium $\{\widetilde{\alpha}, \widetilde{q}\}$ relative to $\mathcal{A}$ is a strongly robust equilibrium relative to any Γ. ■

Theorem 1 shows that the strong robustness of a pure-strategy equilibrium relative to $\mathcal{A}$ is a sufficient condition for it to be strongly robust to any complex mechanisms that assign a contract conditional on agents' reports. This implies that any strongly robust pure-strategy equilibrium relative to $\mathcal{A}$ persists, regardless of what continuation equilibrium agents play upon the deviation of any principal to any complex mechanism. This result is significant, since it provides for the first time a way to identify a strongly robust equilibrium in the multiple-agency case using a very simple game without communication, which can be supported regardless of what strategic interaction between agents is expected upon any principal's deviation to any complex mechanism. It is, however, important to note that Theorem 1 is not applied to a mixed-strategy equilibrium relative to $\mathcal{A}$, as illustrated in the example. Consider the expected probability distribution for principal j over his actions when he deviates to some complex mechanism γ^j, the other principals' contracts are given by α^{-j}, and agents' efforts are e:

$$\acute{\alpha}^j(e) = \int_{\mathcal{C}^j} \gamma^j(c^j)(e) d\widetilde{p}^c(c^j|e, \gamma^j, \alpha^{-j}).$$

If the other principals use pure strategies for their contract choice, then $\acute{\alpha}^j(e)$ is fixed for every $e \in E$. However, when the other principals use mixed strategies, α^{-j} depends on the realization of the contracts according to the principals' mixed strategies. This implies that in general $\acute{\alpha}^j(e)$ changes as the contracts offered by the other principals change, because agents' communication decisions depend on the contracts offered by the other principals. This effectively means that the deviator's action is correlated to the actions of the other principals for any possible effort choices. This type of strategic response is not feasible when a principal is restricted to offering a single contract. Therefore, a principal might be able to find a profitable deviation to a complex mechanism when the other principals use mixed strategies relative to $\mathcal{A}$.[5]

4 Deterministic Single-Contract Games

Section 3 considers the strong robustness of an equilibrium relative to single contracts that allow principals to randomize their actions conditional on agents' efforts. However, much applied research on competing principals often focuses on competition in deterministic contracts. This restriction on the set of single contracts may lead to a loss of generality. When a principal offers a deterministic contract, his action is always deterministic for any given effort chosen by agents. Suppose that a principal deviates to a complex mechanism. Agents might play a mixed-strategy continuation equilibrium for their message and effort choices upon the principal's deviation, which induces a random action for the deviating principal for any efforts that the agents could have chosen even if the mechanism assigns a deterministic contract as a function of agents' messages. Therefore, a new continuation equilibrium might occur upon a principal's deviation to a complex mechanism. This section addresses the question as to when restriction to deterministic contracts makes sense.

In many applications of competition in deterministic contracts, the action

[5]This in fact suggests that mixed-strategy equilibria relative to single contracts may be fragile regarding a principal's deviation to a complex mechanism. One of the incentives for a principal to use a mixed strategy is that he wants to conceal the contract he offers to agents. However, the other principals may use complex mechanisms to identify the contract offered to agents so that they can change their contracts in response to changes in the principal's contract.

of each principal j can be separated with respect to agents and is given by $y^j = (y_1^j, \cdots, y_I^j)$. y_i^j is often the size of the monetary transfer from principal j to agent i. One type of deterministic contract specifies monetary transfer from principal j to agent i as a function of all the agent's efforts. In this case, a deterministic contract offered by principal j to agent i is denoted by $\boldsymbol{a}_i^j : E \to Y_i^j$. Let $\boldsymbol{A}^j = \times_{s=1}^I \boldsymbol{A}_s^j$, $\boldsymbol{A}_i = \times_{t=1}^J \boldsymbol{A}_i^t$ and $\boldsymbol{A} = \times_{t=1}^J \boldsymbol{A}^t$. The other type of deterministic contract specifies the monetary transfer from principal j to agent i as a function of only the effort of agent i. Thus, $a_i^j : E_i \to Y_i^j$ denotes a deterministic contract offered by principal j to agent i. Let $A^j = \times_{s=1}^I A_s^j$, $A_i = \times_{t=1}^J A_i^t$ and $A = \times_{t=1}^J A^t$.

Competition in deterministic contracts has often been employed in the quasi-linear payoff environment where all players are risk-neutral in terms of money. Bernheim and Whinston [3], Klemperer and Meyer [8], Martimort and Stole [10], and Parlour and Rajan [12] modeled the competition in deterministic contracts in the common agency case using such a payoff environment. Prat and Rustichini [18], Weber and Xiong [20], and Strulovici and Weber [19] used the same approach in the multiple-agency case. The two types of quasilinear environments have been considered in applied work, depending on whether there are externalities among agents. The first type of payoff environment is characterized in E1 below. In this environment, there are externalities among agents because an agent's payoffs depend on the efforts of other agents.

E1. The payoff function for each principal i is $v^j(e, y) = h^j(e) - \sum_{s=1}^I y_s^j$ and that for each agent i is $u_i(e, y) = g_i(e) + \sum_{t=1}^J y_i^t$.

Weber and Xiong [20] and Strulovici and Weber [19] considered the single-contract game relative to $\boldsymbol{A}$ in payoff environment E1. Let $r_i : \boldsymbol{A} \to \Delta(E_i)$ denote an effort strategy for agent i in the single-contract offer game relative to $\boldsymbol{A}$. Corollary 1 shows that the strong robustness of a pure-strategy equilibrium relative to $\boldsymbol{A}$ is a sufficient condition for it to be strongly robust relative to any complex mechanisms in payoff environment E1.

Corollary 1 *Any strongly robust pure-strategy equilibrium* $\{\widetilde{\boldsymbol{a}}, \widetilde{r}\}$ *relative to* $\boldsymbol{A}$ *is a strongly robust equilibrium relative to any* Γ *in payoff environment E1.*

For the proof of Corollary 1, we only need to prove that a strongly robust equilibrium relative to $\boldsymbol{A}$ is a strongly robust pure-strategy equilibrium relative to $\mathcal{A}$ in payoff environment E1 and then we can simply apply Theorem 1. In payoff environment E1, players are risk-neutral in terms of money, so players are indifferent between a random contract and a corresponding deterministic contract that specifies the expected transfer from the random contract for any given efforts. Therefore, if all principals offer deterministic contracts in a pure-strategy equilibrium, no new strategic interaction between agents arises upon deviation of any principal j to any random contract α^j.

Prat and Rustichini [18] considered the single-contract game relative to A in payoff environment E2 detailed below. In this environment, there are no externalities among agents.[6]

E2. The payoff function for each principal i is $v^j(e, y) = h^j(e) - \sum_{s=1}^{I} y_s^j$ and that for each agent i is $u_i(e, y) = d_i(e_i) + \sum_{t=1}^{J} y_i^t$.

Let $z_i : A \to \Delta(E_i)$ denote an effort strategy for agent i in the single-contract offer game relative to A. Corollary 2 shows that any pure-strategy equilibrium relative to A is strongly robust relative to any complex mechanisms.

Corollary 2 *Any pure-strategy equilibrium $\{\widetilde{a}, \widetilde{z}\}$ relative to A is a strongly robust equilibrium relative to any Γ in payoff environment E2.*

The following steps are used for the proof of Corollary 2 in the Appendix. First, we prove that in payoff environment E2, any pure-strategy equilibrium relative to A is in fact strongly robust relative to A. Second, we prove that any strongly robust pure-strategy equilibrium relative to A is a strongly robust equilibrium relative to $\mathcal{A}$. Then, Corollary 2 follows immediately by Theorem 1. Corollary 2 implies that any pure-strategy equilibrium relative to A never disappears in payoff environment E2, regardless of what continuation equilibrium agents play upon deviation of any principal to any complex mechanism that assigns a contract conditional on agents' reports. Although

[6]Note that both payoff environments imply that each principal's action can be separated with respect to agents and each agent's payoff can also be separated with respect to principals' actions. Attar et al. [1] took a positive approach and showed that in the common-agency case, only the separability condition for the agent's payoff function is needed for weakly robust equilibrium allocations with the restrictions on strategy spaces often observed in applied research (agents are restricted to use pure strategies and principals to use deterministic mechanisms). The separability condition is weaker than the conditions imposed in E1 and E2. In particular, risk neutrality in terms of money is not needed.

the strong robustness is not considered in the game of Prat and Rustichini, Corollary 2 shows that any pure-strategy equilibrium in the game of Prat and Rustichini survives regardless of what continuation equilibrium (including mixed-strategy continuation equilibria) agents play upon the deviation of any principal to any complex mechanism.

5 Discussion

This paper investigated the strong robustness of an equilibrium relative to single contracts and showed that we can examine single-contract games for strongly robust equilibria in the case of complete information and contractible effort. However, the results for single-contract games in the multiple-agency case are not extended to the environment of non-contractible effort or incomplete information in an obvious way. For example, a principal's single offer becomes simply his (random) action if agents' efforts are not contractible at all. In this case, the realization of each principal's action is independent of agents' effort choices. If a principal unilaterally deviates to a complex mechanism while other principals continue to offer their actions, agents choose messages to send to the deviator along with effort choices. This implies that once a principal deviates to a complex mechanism, he can then correlate his action to agents' effort choices. If a principal simply offers his action, this type of strategic correlation is not possible, so a principal may gain by deviating to a complex mechanism.[7]

Consider the case in which agents' payoff types are private information. If a principal unilaterally deviates to a complex mechanism while other principals continue to offer single contracts, agents' strategies for messages and efforts will depend on their payoff types. These type-dependent strategies mean that the contract of the deviating principal is contingent on agents' payoff types. This sort of contract offer is not feasible if principals simply offer single contracts. Therefore, a new continuation equilibrium may arise upon deviation of a principal to a complex mechanism. Then, it may be argued that we should consider a truth-telling equilibrium relative to direct mechanisms defined over agents' private information for a strongly robust

[7]Attar et al. [2] recently studied the role of a principal's recommendations to agents in the case of non-contractible effort and complete information. Their interest was to identify a class of equilibria relative to single actions with recommendations that are weakly robust to any complex mechanisms in the case of non-contractible effort and complete information.

equilibrium. The beauty of the revelation principle for the case of a single principal is that any continuation equilibrium (including a mixed-strategy continuation equilibrium) for communication for any complex mechanism can be replaced by a (truth-telling) pure-strategy continuation equilibrium for the corresponding direct mechanism. However, this does not hold in the case of multiple principals and multiple agents. If agents play a mixed-strategy continuation equilibrium for communication for a collection of mechanisms offered by multiple principals, they can correlate the principals' actions or contracts for any profile of payoff types. In general, a pure-strategy continuation equilibrium cannot generate this correlation, regardless of whether it is truth-telling or not.[8]

Appendix

Claim 1: Unilateral Deviations

Principal 1's deviation: Note that $\acute{y}^1$ is the strictly dominant action for principal 1 when the agent chooses e. Therefore, we only consider deviation to a contract α^1, with $\alpha^1(e) = 0$. Suppose that principal 1 deviates to some contract $\alpha^1 \in \mathcal{A}^1$, with $\alpha^1(e) = 0$, while principal 2 continues to offer $\bar{\alpha}^2$. If the agent chooses e, her payoff is $8\alpha^1(e) + 5(1 - \alpha^1(e)) = 5$. If the agent chooses e', her payoff is $7\alpha^1(e') + 2(1 - \alpha^1(e')) = 2 + 5\alpha^1(e')$. Therefore, the agent's optimal effort choice is:

$$\widetilde{q}(\alpha^1, \bar{\alpha}^2) = \begin{cases} 0 & \text{if } \alpha^1(e') > \frac{3}{5} \\ z & \text{if } \alpha^1(e') = \frac{3}{5} \\ 1 & \text{if } \alpha^1(e') < \frac{3}{5} \end{cases}$$

where z is any number in $[0, 1]$. Suppose that principal 1 deviates to a contract α^1 such that $\alpha^1(e') < \frac{3}{5}$. Then the agent chooses e for sure, principal 2 chooses y^2 given his contract $\bar{\alpha}^2$, and principal 1's payoff is $2\alpha^1(e) + 3(1 - \alpha^1(e)) = 3$. Thus, there is no continuation that makes principal 1 strictly better off if he deviates to any contract α^1 such that $\alpha^1(e') < \frac{3}{5}$. Suppose that principal 1 deviates to a contract α^1 such that $\alpha^1(e') > \frac{3}{5}$. Then the agent choose e' for sure, principal 2 chooses $\acute{y}^2$ given his contract $\bar{\alpha}^2$, and

[8] Epstein and Peters [6] and Yamashita [21] each considered the different classes of direct mechanisms and showed that truth-telling equilibria relative to these direct mechanisms support the equilibrium allocations relative to any complex mechanisms for which agents play a pure-strategy continuation equilibrium for any collection of mechanisms.

principal 1's payoff is $3\alpha^1(e) + 2(1 - \alpha^1(e)) = 2$, which is less than 3. Thus, there is no continuation that makes principal 1 strictly better off if he deviates to any contract α^1 such that $\alpha^1(e') > \frac{3}{5}$. Suppose that principal 1 deviates to a contract α^1 such that $\frac{3}{5} = \alpha^1(e')$. Then it is optimal for the agent to choose e with probability z and e' with probability $1-z$ for any $z \in [0,1]$, and principal 1's payoff is $z(3 - \alpha^1(e)) + (1-z)(2 + \alpha^1(e)) = 2 + z \leq 3$. Thus, there is no continuation equilibrium that makes principal 1 strictly better when he deviates to any contract α^1 such that $\frac{3}{5} = \alpha^1(e')$. We conclude that there is no continuation equilibrium that makes principal 1 strictly better off upon his deviation to any random contract in $\mathcal{A}^1$.

Principal 2's deviation: Note that y^2 is the strictly dominant action for principal 2 when the agent chooses e. Therefore, we only consider deviation to a contract α^2, with $\alpha^2(e) = 1$. Suppose that principal 2 deviates to some contract $\alpha^2 \in \mathcal{A}^2$, with $\alpha^2(e) = 1$, and principal 1 has drawn $\hat{\alpha}^1$ from $\widetilde{\tau}^1$. If the agent chooses e, her payoff is $5\alpha^2(e) + 4(1 - \alpha^2(e)) = 5$. If the agent chooses e', her payoff is $6\alpha^2(e') + 2(1 - \alpha^2(e')) = 2 + 4\alpha^2(e')$. Therefore, the agent's optimal effort choice at $(\hat{\alpha}^1, \alpha^2)$ is:

$$\widetilde{q}(\hat{\alpha}^1, \alpha^2) = \begin{cases} 0 & \text{if } \alpha^2(e') > \frac{3}{4} \\ z' & \text{if } \alpha^2(e') = \frac{3}{4} \\ 1 & \text{if } \alpha^2(e') < \frac{3}{4} \end{cases}$$

where z' is any number in $[0,1]$. Therefore, principal 2's payoff on $(\hat{\alpha}^1, \alpha^2)$ is:

$$V^2(\hat{\alpha}^1, \alpha^2, \widetilde{q}) = \begin{cases} 3 + 3\alpha^2(e') & \text{if } \alpha^2(e') > \frac{3}{4} \\ \frac{21}{4} - \frac{9}{4}z' & \text{if } \alpha^2(e') = \frac{3}{4} \\ 3 & \text{if } \alpha^2(e') < \frac{3}{4} \end{cases} \tag{2}$$

Suppose that principal 2 deviates to some contract $\alpha^2 \in \mathcal{A}^2$, with $\alpha^2(e) = 1$, and principal 1 has drawn $\ddot{\alpha}^1$ from $\widetilde{\tau}^1$. If the agent chooses e, her payoff is $5\alpha^2(e) + 4(1 - \alpha^2(e)) = 5$. If the agent chooses e', her payoff is $3\alpha^2(e') + 7(1 - \alpha^2(e')) = 7 - 4\alpha^2(e')$. Therefore, the agent's optimal effort choice at $(\ddot{\alpha}^1, \alpha^2)$ is:

$$\widetilde{q}(\ddot{\alpha}^1, \alpha^2) = \begin{cases} 0 & \text{if } \alpha^2(e') < \frac{1}{2} \\ z'' & \text{if } \alpha^2(e') = \frac{1}{2} \\ 1 & \text{if } \alpha^2(e') > \frac{1}{2} \end{cases}$$

where z'' is any number in $[0,1]$. Therefore, principal 2's payoff at $(\ddot{\alpha}^1, \alpha^2)$

is:

$$V^2(\ddot{\alpha}^1, \alpha^2, \widetilde{q}) = \begin{cases} 6 - 4\alpha^2(e') & \text{if } \alpha^2(e') < \frac{1}{2} \\ 4 - z'' & \text{if } \alpha^2(e') = \frac{1}{2} \\ 3 & \text{if } \alpha^2(e') > \frac{1}{2} \end{cases} \tag{3}$$

Now we determine the values of p such that, given p, principal 2 cannot gain in any continuation equilibrium upon his deviation to any contract in $\mathcal{A}^2$.

Case 1: Suppose that principal 2 deviates to a contract α^2 such that $\alpha^2(e') < \frac{1}{2}$. Principal 2's payoff is:

$$V^2(\widetilde{\tau}^1, \alpha^2, \widetilde{q}) = 3p + (1-p)\left(6 - 4\alpha^2(e')\right) = 6 - 3p - (1-p)4\alpha^2(e').$$

Since $V^2(\widetilde{\tau}^1, \alpha^2, \widetilde{q})$ cannot be greater than $6 - 3p$, principal 2 cannot gain, regardless of the value of p, by deviating to any α^2 such that $\alpha^2(e') < \frac{1}{2}$.

Case 2: Suppose that principal 2 deviates to a contract α^2 such that $\alpha^2(e') = \frac{1}{2}$. Then, principal 2's payoff is:

$$V^2(\widetilde{\tau}^1, \alpha^2, \widetilde{q}) = 3p + (1-p)(4 - z'').$$

$V^2(\widetilde{\tau}^1, \alpha^2, \widetilde{q})$ is maximized when $z'' = 0$ and it is equal to $4 - p$. Since $6 - 3p \geq 4 - p$ for all $p \in [0, 1]$, principal 2 cannot gain in any continuation equilibrium upon his deviation to any contract α^2 such that $\alpha^2(e') = \frac{1}{2}$.

Case 3: Suppose that principal 2 deviates to a contract α^2 such that $\frac{1}{2} < \alpha^2(e') < \frac{3}{4}$.

$$V^2(\widetilde{\tau}^1, \alpha^2, \widetilde{q}) = 3.$$

Since $6 - 3p \geq 3$, principal 2 cannot gain, regardless of the value of p, by deviating to any α^2 such that $\frac{1}{2} < \alpha^2(e') < \frac{3}{4}$.

Case 4: Suppose that principal 2 deviates to a contract α^2 such that $\alpha^2(e') = \frac{3}{4}$. Then principal 2's payoff is:

$$V^2(\widetilde{\tau}^1, \alpha^2, \widetilde{q}) = p\left(\frac{21}{4} - \frac{9}{4}z'\right) + 3(1-p).$$

$V^2(\widetilde{\tau}^1, \alpha^2, \widetilde{q})$ is maximized when $z' = 0$ and it is equal to $3 + \frac{9}{4}p$. Therefore, it is not profitable for principal 2 to deviate any contract such that $\alpha^2(e') = \frac{3}{4}$ in any continuation equilibrium upon his deviation if $6 - 3p \geq 3 + \frac{9}{4}p$, i.e., $p \leq \frac{4}{7}$.

Case 5: Suppose that principal 2 deviates to a contract α^2 such that $\alpha^2(e') > \frac{3}{4}$. Then principal 2's payoff is:

$$V^2(\widetilde{\tau}^1, \alpha^2, \widetilde{q}) = p\left(3 + 3\alpha^2(e')\right) + 3(1-p).$$

Among the contracts that satisfy $\alpha^2(e') > \frac{3}{4}$, the contract α^2 with $\alpha^2(e') = 1$ creates the highest payoff for principal 2 and is equal to $3 + 3p$. Therefore, principal 2 cannot gain by deviating to any contract such that $\alpha^2(e') > \frac{3}{4}$ if $6 - 3p \geq 3 + 3p$, i.e., $p \leq 1/2$.

From cases 1–5, we can conclude that if $p \in [0, \frac{1}{2}]$, then principal 2 cannot gain by deviating to any contract in $\mathcal{A}^2$, regardless of what continuation equilibrium the agent plays upon his deviation. Since there is no continuation equilibrium that makes any principal strictly better off either on the equilibrium path or off the equilibrium path following his unilateral deviation to any random contract, $\{\widetilde{\tau}^1, \widetilde{\tau}^2\}$ is the profile of strongly robust equilibrium strategies for principals relative to $\mathcal{A}$ for any $p \in [0, 1/2]$.

Proof of Corollary 1. Since Theorem 1 holds, we only need to show that $\{\widetilde{\boldsymbol{a}}, \widetilde{r}\}$ is a strongly robust pure-strategy equilibrium relative to $\mathcal{A}$. Since $\{\widetilde{\boldsymbol{a}}, \widetilde{r}\}$ is a strongly robust equilibrium relative to $\boldsymbol{A}$ and $\widetilde{\boldsymbol{a}}^j$ is in $\mathcal{A}^j$ for each j, there is no continuation equilibrium that makes any principal strictly better off on the equilibrium path in the game relative to $\mathcal{A}$. Therefore, it is sufficient to show that there is no continuation equilibrium upon deviation of any principal j to any contract in $\mathcal{A}^j$ that makes the deviating principal strictly better off. Suppose that principal j unilaterally deviates to some contract α^j in $\mathcal{A}^j$. Let $\widetilde{p} = \{\widetilde{p}_s\}_{s=1}^I$ characterize an arbitrary continuation equilibrium at $(\alpha^j, \widetilde{\boldsymbol{a}}^{-j})$. Thus, $\widetilde{p}_i(\cdot|\alpha^j, \widetilde{\boldsymbol{a}}^{-j}) \in \Delta(E_i)$ is the equilibrium probability distribution that agent i uses for her effort choice at $(\alpha^j, \widetilde{\boldsymbol{a}}^{-j})$. Let $\acute{\boldsymbol{a}}_i^j(e) = \int_{Y_i^j} y_i^j d\alpha^j(e)$ for all i. Then the payoff for agent i becomes:

$$U_i(\widetilde{p}_i, \widetilde{p}_{-i}, \alpha^j, \widetilde{\boldsymbol{a}}^{-j}) =$$
$$\int_E \left(g_i(e) + \sum_{t \neq j} \widetilde{\boldsymbol{a}}_i^t(e)\right) d\widetilde{p}(e|\alpha^j, \widetilde{\boldsymbol{a}}^{-j}) + \int_E \left(\int_{Y_i^j} y_i^j d\alpha^j(e)\right) d\widetilde{p}(e|\alpha^j, \widetilde{\boldsymbol{a}}^{-j}) =$$
$$\int_E \left(g_i(e) + \sum_{t \neq j} \widetilde{\boldsymbol{a}}_i^t(e) + \acute{\boldsymbol{a}}_i^j(e)\right) d\widetilde{p}(e|\alpha^j, \widetilde{\boldsymbol{a}}^{-j}) =$$
$$U_i(\widetilde{p}_i, \widetilde{p}_{-i}, \acute{\boldsymbol{a}}^j, \widetilde{\boldsymbol{a}}^{-j}).$$

where $\acute{\boldsymbol{a}}^j = (\acute{\boldsymbol{a}}^j_1, \acute{\boldsymbol{a}}^j_2, \cdots \acute{\boldsymbol{a}}^j_I)$. Since $U_i(\widetilde{p}_i, \widetilde{p}_{-i}, \alpha^j, \widetilde{\boldsymbol{a}}^{-j}) = U_i(\widetilde{p}_i, \widetilde{p}_{-i}, \acute{\boldsymbol{a}}^j, \widetilde{\boldsymbol{a}}^{-j})$ and $\{\widetilde{p}_s\}_{s=1}^I$ characterizes a continuation equilibrium at $(\alpha^j, \widetilde{\boldsymbol{a}}^{-j})$, the effort decision for agent i, $\widetilde{p}_i$, must satisfy, for all $\acute{e}_i \in E_i$:

$$U_i(\widetilde{p}_i, \widetilde{p}_{-i}, \acute{\boldsymbol{a}}^j, \widetilde{\boldsymbol{a}}^{-j}) \geq U_i(\acute{e}_i, \widetilde{p}_{-i}, \acute{\boldsymbol{a}}^j, \widetilde{\boldsymbol{a}}^{-j}).$$

This implies that $\{\widetilde{p}_s\}_{s=1}^I$ is a continuation equilibrium when the collection of single-incentive contracts is $(\acute{\boldsymbol{a}}^j, \widetilde{\boldsymbol{a}}^{-j})$.

The payoff for principal j upon his unilateral deviation to some α^j is:

$$\begin{gathered} V^j(\alpha^j, \widetilde{\boldsymbol{a}}^{-j}, \widetilde{p}) = \\ \int_E \left[h^j(e) - \int_{Y^j} \left(\sum_{s=1}^I y_s^j \right) d\alpha^j(e) \right] d\widetilde{p}(e|\alpha^j, \widetilde{\boldsymbol{a}}^{-j}) = \\ \int_E \left[h^j(e) - \sum_{s=1}^I \int_{Y_s^j} y_s^j d\alpha^j(e) \right] d\widetilde{p}(e|\alpha^j, \widetilde{\boldsymbol{a}}^{-j}) = \\ \int_E \left(h^j(e) - \sum_{s=1}^I \acute{\boldsymbol{a}}_s^j(e_s) \right) d\widetilde{p}(e|\alpha^j, \widetilde{\boldsymbol{a}}^{-j}) = \\ V^j(\acute{\boldsymbol{a}}^j, \widetilde{\boldsymbol{a}}^{-j}, \widetilde{p}). \end{gathered} \tag{4}$$

The second equality holds because the expected sum of monetary transfers is equal to the sum of expected monetary transfers for any given $e \in E$, i.e., $\int_{Y^j} \left(\sum_{s=1}^I y_s^j \right) d\alpha^j(e) = \sum_{s=1}^I \int_{Y_s^j} y_s^j d\alpha^j(e)$ for any given $e \in E$. Because $\{\widetilde{\boldsymbol{a}}, \widetilde{r}\}$ is a strongly robust equilibrium relative to $\boldsymbol{A}$, we have $V^j(\widetilde{\boldsymbol{a}}^j, \widetilde{\boldsymbol{a}}^{-j}, \widetilde{r}) \geq V^j(\acute{\boldsymbol{a}}^j, \widetilde{\boldsymbol{a}}^{-j}, \widetilde{p}) = V^j(\alpha^j, \widetilde{\boldsymbol{a}}^{-j}, \widetilde{p})$. Therefore, it is not profitable for principal j to deviate to any random contract, regardless of what continuation equilibrium agents play upon his deviation. Therefore, any strongly robust pure-strategy equilibrium $(\widetilde{\boldsymbol{a}}, \widetilde{r})$ relative to $\boldsymbol{A}$ is a strongly robust pure-strategy equilibrium relative to $\mathcal{A}$. Corollary 1 follows immediately by Theorem 1.

Proof of Corollary 2: First, we prove that any pure-strategy equilibrium $(\widetilde{a}, \widetilde{z})$ relative to A is strongly robust relative to A. Suppose that it is not. Then, for some principal j, there must exist an array of incentive contracts and a continuation equilibrium (a^j, z) such that:

$$V^j(a^j, \widetilde{a}^{-j}, z) > V^j(\widetilde{a}^j, \widetilde{a}^{-j}, \widetilde{z}). \tag{5}$$

Let $z_i(\cdot|a^j, \widetilde{a}^{-j})$ be the equilibrium probability distribution for agent i on E_i at $(a^j, \widetilde{a}^{-j})$. Let $\overline{e} = (\overline{e}_1, \cdots, \overline{e}_I)$ be an arbitrary array of efforts in $\times_{s=1}^{I} \mathsf{supp}\ z_s(\cdot|a^j, \widetilde{a}^{-j})$ such that:

$$h^j(\overline{e}) - \sum_{s=1}^{I} a_s^j(\overline{e}_s) \geq \tag{6}$$

$$V^j(a^j, \widetilde{a}^{-j}, z) = \int_E \left(h^j(e) - \sum_{s=1}^{I} a_s^j(e_s) \right) dz(e|a^j, \widetilde{a}^{-j}),$$

where $z(e|a^j, \widetilde{a}^{-j}) = \Pi_{s=1}^{I} z_s(e_s|a^j, \widetilde{a}^{-j})$. We construct a single contract $\acute{a}_i^j$ for agent i such that:

$$\acute{a}_i^j(e_i) = \begin{cases} a_i^j(e_i) & \text{if } e_i = \overline{e}_i \\ a_i^j(e_i) - \epsilon & \text{otherwise} \end{cases} \tag{7}$$

where ϵ is a very small positive number.

Suppose that principal j offers $\acute{a}^j = (\acute{a}_1^j, \cdots, \acute{a}_I^j)$, while the other principals still offer the equilibrium contracts $\widetilde{a}^{-j}$. Consider the optimal choice for agent i at $(\acute{a}^j, \widetilde{a}^{-j})$. Since $\overline{e}_i$ is in $\mathsf{supp}\ z_i(\cdot|a^j, \widetilde{a}^{-j})$, the definition of $\acute{a}_i^j$ given in (7) implies that, for all $e_i \neq \overline{e}_i$:

$$d_i(\overline{e}_i) + \sum_{t \neq j} \widetilde{a}_i^t(\overline{e}_i) + \acute{a}_i^j(\overline{e}_i) \geq d_i(e_i) + \sum_{t \neq j} \widetilde{a}_i^t(e_i) + a_i^j(e_i). \tag{8}$$

The definition of $\acute{a}_i^j$ given in (7) also implies that, for all $e_i \neq \overline{e}_i$:

$$d_i(e_i) + \sum_{t \neq j} \widetilde{a}_i^t(e_i) + a_i^j(e_i) > d_i(e_i) + \sum_{t \neq j} \widetilde{a}_i^t(e_i) + \acute{a}_i^j(e_i). \tag{9}$$

Equations (8) and (9) imply that a continuation equilibrium for $(\acute{a}^j, \widetilde{a}^{-j})$ is unique and that each agent i chooses $\overline{e}_i$. Therefore, we have:

$$V^j(\acute{a}^j, \widetilde{a}^{-j}, \widetilde{z}) = h^j(\overline{e}) - \sum_{s=1}^{I} a_s^j(\overline{e}_s).$$

Furthermore, (6) implies that:

$$V^j(\acute{a}^j, \widetilde{a}^{-j}, \widetilde{z}) = h^j(\overline{e}) - \sum_{s=1}^{I} a_s^j(\overline{e}_s) \geq V^j(a^j, \widetilde{a}^{-j}, z). \tag{10}$$

From (5) and (10), we have $V^j(\acute{a}^j, \widetilde{a}^{-j}, \widetilde{z}) > V^j(\widetilde{a}^j, \widetilde{a}^{-j}, \widetilde{z})$. This contradicts the assumption that $\{\widetilde{a}, \widetilde{z}\}$ is a pure-strategy equilibrium relative to A. Therefore, any pure-strategy equilibrium relative to A must be strongly robust relative to A. We can prove that any strongly robust pure-strategy equilibrium relative to A is strongly robust relative to $\mathcal{A}$ by following the general logic used in the proof of Corollary 1. Then Corollary 2 follows immediately by Theorem 1.

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